Electrostatics in vacuum.

Homework

1. Sketch the diagram of the field lines for:

   (a) two equal positive charges q at some distance from each other;

   (b) charges $2q$ and $-q$, some distance away from each other.

   [Hint: start by thinking about sketching the field lines near each of the charges and far away from them; then connect these systems of lines in a meaningful way.]

2. Electric charge is distributed with constant density σ on the surface of a disc of radius a. Show that the potential at distance z away from the disc along the axis of symmetry is

$$\phi(z) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - z\right)$$

   Hence, calculate the electric field on the axis, and verify that

$$E = \frac{\sigma}{2\varepsilon_0} \quad (\text{for } z \ll a), \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{\pi a^2\sigma}{z^2} \quad (\text{for } z \gg a),$$

   i.e., the field of an infinite charged plane and the field of point charge $q = \pi a^2\sigma$, respectively.

   [Hint: Use the general formula

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dS'$$

   Choose the origin O at the centre of the disc. The z-axis is normal to the disc, and the disc is in the x-y plane, so that $\boldsymbol{r}' = (x', y', 0)$ for a point on the disc, and $\boldsymbol{r} = (0, 0, z)$, since we evaluate the potential on the x-axis. Using

$$|\boldsymbol{r}' - \boldsymbol{r}| = \sqrt{x'^2 + y'^2 + z^2}$$

   with z held fixed, do the integral over the disc by changing to polar coordinates $(x', y') \to (\rho, \psi)$, for which $dS' = dx'dy' \to \rho d\rho d\psi$. Use $\boldsymbol{E} = E_z\boldsymbol{k} = -\partial\phi/\partial z\boldsymbol{k}$ to evaluate the electric field along the z-axis.]

3. (a) Starting from Coulomb's law, obtain an expression for the electric field $\boldsymbol{E}$ at the position $\boldsymbol{r}$ due to a set of charges q_i $(i = 1, \ldots, n)$ at positions $\boldsymbol{r}_i$.

   (b) State and prove Gauss's law for this set of charges. You may assume the result for the solid angle subtended by a closed surface.

   (c) Show that the solid angle subtended by a circular disc at a point on its axis is

$$\Omega = 2\pi(1 - \cos\theta_0)$$

   where θ_0 is the angle between the axis and a line joining the edge of the disc to the point.

   (d) Two charges q and q' are placed on the axis of a circular disc of radius a and to the same side of the disc. If q and q' are respectively $2a$ and $3a$ from the centre of the disc, then find q' in terms of q if the total flux of the electric field through the disc is zero.

   [Exam 2004 Question 1]

4. (a) Show that the spherically symmetric charge distribution

$$\rho(r) = \frac{q}{4\pi a^2}\frac{e^{-r/a}}{r}, \tag{1}$$

where q and a are constants, gives the electrostatic potential

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}(1 - e^{-r/a}). \tag{2}$$

[Hint : Assume that $\boldsymbol{E} = E(r)\hat{\boldsymbol{r}}$ and apply Gauss's law to a sphere of radius r. Find the electric field and integrate it to get the potential.]

(b) By integrating $\rho(r)$ over the whole space show that q is the total charge. Hence, show that $\phi(r)$ from equation (2) has the correct behavious at $r \gg a$.
[Hint: use spherical shells with the volume element $dV = 4\pi r^2 dr$ for integration.]

5. Electric charge is distributed on an infinite straight line at constant linear density λ (charge per unit length of the line). Apply Gauss's law to a cylinder of length l and radius ρ, coaxial with the line, and deduce that the the electric field at distance ρ from the line is

$$E(\rho) = \frac{\lambda}{2\pi\varepsilon_0\rho}.$$

Hence, show that the potential is

$$\phi(\rho) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + \text{const.}$$

How does the potential behave as $\rho \to \infty$?

6. Two spherical conducting shells of radii a and b, where $b > a$, are arranged concentically and are charged to potenials ϕ_a and ϕ_b respectively. Show that the potential ϕ at points between the shells and at points $r > b$ is given by

$$\phi(r) = \frac{ab(\phi_a - \phi_b)}{b-a}\frac{1}{r} + \frac{b\phi_b - a\phi_a}{b-a} \qquad (a < r < b),$$

$$\phi(r) = \frac{b\phi_b}{r} \qquad (r > b).$$

[Hint : Use spherical polar coordinates, assume $\phi = \phi(r)$ and solve Laplace's equation in the two regions. Identify the constants of integration from the boundary conditions. The Laplacian in spherical polar coordinates (r, θ, ψ) is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2},$$

where we use ψ for the azimuthal angle, instead of the usual ϕ, to avoid confusion with the potential.]

7. The potential ϕ due to a distribution of charge $\rho(\boldsymbol{r})$ confined to a finite volume V surrounding the origin is

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dV'.$$

(a) We are interested in how $\phi(\boldsymbol{r})$ behaves for $r \gg r'$. Using the binomial expansion of $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ show that the terms up to $1/r^3$ can be written as

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{p}}{r^3} + \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\frac{x_i x_j}{r^5}Q_{ij} + \dots\right)$$

where

$$Q = \int_V \rho(\boldsymbol{r})dV$$

is the total charge of the distribution,

$$\boldsymbol{p} = \int_V \boldsymbol{r}\rho(\boldsymbol{r})dV$$

is the dipole moment of the distribution,

$$Q_{ij} = \int_V \left(3x_i x_j - \delta_{ij}r^2\right)\rho(\boldsymbol{r})dV$$

is the quadrupole moment tensor, and (x_1, x_2, x_3) are the Cartesian components of $\boldsymbol{r}$.

(b) If the total charge Q is zero, show that the dipole moment $\boldsymbol{p}$ is independent of the choice origin of coordinates. [Hint: find $\boldsymbol{p}$ with respect to a new origin at $\boldsymbol{a}$.]

(c) If the distribution ρ consists only of two equal and opposite point charges, $q > 0$ and $-q$, separated by the vector $\boldsymbol{l}$ (directed from $-q$ to q), show that the dipole moment $\boldsymbol{p}$ defined above reproduces the usual result $\boldsymbol{p} = q\boldsymbol{l}$. [Hint: write $\rho(\boldsymbol{r})$ for the two charges using the Dirac δ-function.]

(d) The distribution ρ consists of three point charges $-q$, $+2q$ and $-q$, which lie on the z axis at points $(0,0,-a)$, $(0,0,0)$ and $(0,0,a)$, respectively. Show that $Q = 0$ and $\boldsymbol{p} = 0$ and find the components of Q_{ik}.

Hint for part (a): use the binomial expansion to second order,

$$\begin{aligned}|\boldsymbol{r}-\boldsymbol{r}'|^{-1} &= (r^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}' + r'^2)^{-1/2}\\ &= r^{-1}(1+\alpha)^{-1/2}\\ &= r^{-1}\left(1 - \frac{1}{2}\alpha + \frac{3}{8}\alpha^2 + \dots\right)\end{aligned}$$

where $\alpha = r^{-2}(-2\boldsymbol{r}\cdot\boldsymbol{r}' + r'^2)$, keeping terms up to $\sim (r'/r)^2$.

Also, in order to separate the components of $\boldsymbol{r}$ and $\boldsymbol{r}'$ in the integral, taking those of $\boldsymbol{r}$ outside, use

$$\boldsymbol{r}\cdot\boldsymbol{r}' = \sum_{i=1}^{3} x_i x_i' \qquad \text{and} \qquad r^2 = \sum_{i=1}^{3} x_i x_i = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \delta_{ij}.$$

Further Examples

1. Electric charge is distributed with variable density σ on an infinite plane. P is a point distance a from the plane, and dS is a surface element whose distance from P is R.

   (a) Prove that the electric field at P has a component *normal* to the plane, equal to

   $$\frac{a}{4\pi\varepsilon_0}\int\frac{\sigma}{R^3}dS$$

   wher the integral is over the whole plane.

   (b) For constant σ show that this gives a value $\sigma/2\varepsilon_0$ and deduce that in such a case one half of the field arises from those points of the plane that are less than $2a$ from P.

   [Hint: The general formula is

   $$\boldsymbol{E}(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\int_S\frac{\sigma(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\boldsymbol{r}-\boldsymbol{r}')dS'.$$

   In our case $\boldsymbol{r}$ is the position vector P, $\boldsymbol{r'}$ is the position vector of the element dS in the plane, and $R=|\boldsymbol{r}-\boldsymbol{r}'|$. Choose the origin O by dropping a perpendicular from the point P on to the plane. The z-axis is from O through P. Then $\boldsymbol{r'}=(x',y',0)$, since it points to the charge on the plane, $\boldsymbol{r}=(0,0,a)$ since we evaluate the field at P. Consider E_z.

   The surface integral, when converted to polar coordinates (r',ψ'), can, by a change of variable, be written in terms of R where $R^2={r'}^2+a^2$.]

2. A fixed circle of radius a has a charge q placed at a distance $3a/4$ from its centre, on a line through the centre perpendicular to the plane of the circle. Show that the flux of $\boldsymbol{E}$ through the circle is $q/5\varepsilon_0$. If a second charge q', similarly placed at a distance $5a/12$ on the opposite side of the circle, results in no net flux through the circle, show that $q'=13q/20$.

   [Hint: Determine the solid angle subtended by the circle at the charges.]

3. An infinite plane slab of thickness d is filled with charge of uniform density ρ (i.e., $\rho=$ const), with no charge elsewhere. Find the electric field $\boldsymbol{E}$ and show that the corresponding potential is

   $$\phi(x)=\begin{cases}-\dfrac{\rho x^2}{2\varepsilon_0} & \text{for}\quad -\dfrac{d}{2}\le x\le\dfrac{d}{2},\\ -\dfrac{\rho d|x|}{2\varepsilon_0}+\dfrac{\rho_0 d^2}{8\varepsilon_0} & \text{for}\quad |x|\ge\dfrac{d}{2},\end{cases}$$

   where x is measured from the mid-plane of the slab along the normal.

   [Hint: By symmetry, the electric field will be normal to the slab, $E=E(x)$, and at the middle of the slab $E=0$. Apply Gauss's law to a cylinder placed symmetrically with respect to the mid-plane, with bases parallel to it. There are two cases: the bases are either within the slab, or outside. Find the electric field and integrate to obtain the potential.]

4. Two infinitely long cylindrical conducting shells of radii a and b, where $a < b$, are arranged coaxially and are charged to potentials ϕ_a and ϕ_b respectively. Show that the potential between the cylinders is given by

$$\phi(\rho) = \frac{\phi_b - \phi_a}{\ln b - \ln a} \ln \rho + \frac{\phi_a \ln b - \phi_b \ln a}{\ln b - \ln a} \qquad (a < \rho < b)$$

where ρ is the perpendicular distance from the axis of the cylinders.

[Hint : Use cylindrical polar coordinates (ρ, ψ, z) in which the Laplacian given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \psi^2} + \frac{\partial^2}{\partial z^2}.$$

Assume $\phi = \phi(\rho)$ and solve Laplace's equation. Identify the constants of integration from the boundary conditions.]

5. By considering the electric charge on the surface of a conductor to lie within a very thin layer of thickness d, show that there is an outward force on the surface of magnitude

$$\frac{\sigma^2}{2\varepsilon_0}$$

per unit area, where σ is the surface density of charge.

6. A system of charges generates the spherically symmetric electrostatic potential

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}(1 + br)\exp(-2br)$$

where q and b are constants. Show that the sources of the potential are a point charge q located at $r = 0$, and a volume distribution of density

$$\rho(r) = -\frac{qb^3}{\pi}\exp(-2b).$$

[Hint: Use Poisson's equation to calculate $\rho(r)$.]

7. A system of charges consists of $+2q$ at the origin and $-q$ at the two points $(0, 0, \pm a)$, i.e., on the z-axis. Show that at distances $r \gg a$ the potential may be written in the approximate form

$$\phi = \frac{qa^2}{4\pi\varepsilon_0 r^3}(1 - 3\cos^2\theta),$$

where θ is the polar angle of the field point.

Explain in a simple way why $\phi > 0$ in the direction perpendicular to the z axis ($\theta = \pi/2$).