Electromagnetic Theory AMA3001 — Problem sheet 2

Dielectrics.

Homework

1. (a) i. By modelling the polarization of a dielectric by a dipole moment per unit volume, $\boldsymbol{P}$, show that the polarization can be represented by a volume charge density, ρ_p, and a surface charge density, σ_p, given by

$$\rho_p = -\nabla \cdot \boldsymbol{P} \qquad \text{and} \qquad \sigma_p = \boldsymbol{P} \cdot \hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{n}}$ is an outward normal to the surface.

ii. Hence derive the equation $\nabla \cdot \boldsymbol{D} = \rho$, giving the definition of $\boldsymbol{D}$. You may assume the differential form of Gauss's law.

iii. Explain what is meant by a linear, isotropic dielectric?

(b) i. A condenser consists of concentric conducting spheres of radii a and b with $b > a$. The space between the spheres is filled by a dielectric of constant κ. Show that the capacitance of the condenser is

$$C = 4\pi\varepsilon_0 \frac{\kappa ab}{b-a} \tag{1}$$

where ε_0 is the permittivity of free space.

ii. Show that for $b - a \ll a, b$ the capacitance (1) can be approximated by the capacitance of a parallel plate capacitor $C = \varepsilon_0 \kappa A/d$, where A is the area of each plate and d is the spacing between them.

[Similar to Exam 2008 Question 1]

2. The space between two concentric conducting spheres of inner and outer radii a and b is half-filled by a hemispherical shell of material of dielectric constant κ. The rest of the space is vacuum. The inner conductor carries charge Q.

(a) Find the electric field everywhere between the spheres.

(b) Show that the surface charge densities on the part of the inner conductor in contact with the dielectric, and on that in contact with vacuum, are, respectively,

$$\frac{\kappa Q}{2\pi a^2(\kappa+1)} \qquad \text{and} \qquad \frac{Q}{2\pi a^2(\kappa+1)}.$$

(c) Show that the polarization charge density on the surface of the dielectric at $r = a$ is

$$\frac{(1-\kappa)Q}{2\pi a^2(\kappa+1)}.$$

(d) Show that the potential difference between the conductors is

$$V = \frac{Q}{2\pi\varepsilon_0(\kappa+1)}\left(\frac{1}{a} - \frac{1}{b}\right).$$

(e) Hence, find the capacitance of the system and show that it is equal to the sum of the capacitances of two hemispherical capacitors, one with the dielectric, and the other with vacuum inside. [This answer could be expected, since the system can be thought of as two such capacitors connected in parallel.]

3. A spherical condenser consists of two concentric spherical conducting shells of radii a and d. Concentric with these and lying between them is a spherical shell of linear, isotropic dielectric of permittivity ε bounded by the spherical surfaces at $r = b$ and $r = c$ with $a < b < c < d$.

   (a) Show that the capacitance C of the condenser satisfies

   $$\frac{4\pi\varepsilon_0}{C} = \frac{1}{a} - \frac{1}{d} + \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon}\right)\left(\frac{1}{b} - \frac{1}{c}\right).$$

   (b) Show that the polarisation vector between $r = b$ and $r = c$ is

   $$\left(\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\right)\frac{Q}{r^2}\hat{\boldsymbol{r}},$$

   where Q is the charge on the inner conducting shell.

   (c) Find the volume and surface polarization charge densities.

   [Exam 2003 Question 1 – non-bookwork part]

4. A cylindrical coaxial cable has a compound dielectric. The cylindrical conductors are separated by two dielectrics. The inner conductor has an outside radius a. This is surrounded by a dielectric sheath of dielectric constant κ_1 and of outer radius b. Next comes another dielectric sheath of dielectric constant κ_2 and outer radius c.

   If a potential difference V is imposed between the conductors, show that, in cylindrical polar coordinates (ρ, θ, z), the polarization at each point in the two dielectrics is given by

   $$\boldsymbol{P}_1 = \frac{\varepsilon_0(\kappa_1 - 1)\kappa_2 V}{\kappa_2 \ln\dfrac{b}{a} + \kappa_1 \ln\dfrac{c}{b}}\,\frac{\hat{\boldsymbol{\rho}}}{\rho} \qquad \text{and} \qquad \boldsymbol{P}_2 = \frac{\varepsilon_0(\kappa_2 - 1)\kappa_1 V}{\kappa_2 \ln\dfrac{b}{a} + \kappa_1 \ln\dfrac{c}{b}}\,\frac{\hat{\boldsymbol{\rho}}}{\rho}.$$

5. Two parallel conducting plates are separated by a small distance d and maintained at potential difference $\Delta\phi > 0$. A dielectric slab, of dielectric constant κ and of uniform thickness d, is inserted snugly between the plates; however, the slab does not completely fill the volume between the plates.

   (a) Find the electric field in the dielectric and in the vacuum region between the plates.

   (b) Find the surface charge density on that part of the plate (at higher potential) in contact with the dielectric (σ_d), and on its part in contact with the vacuum (σ_v).

   (c) Find the polarization charge density σ_p on the surface of the dielectric slab in contact with the plate at higher potential.

   (d) Show that $\sigma_d + \sigma_p = \sigma_v$.

Further Examples

1. (a) A dielectric has a dipole moment $\boldsymbol{P}$ per unit volume. Show that this is equivalent to a volume charge density ρ_p and a surface charge density σ_p given by

$$\rho_p = -\nabla \cdot \boldsymbol{P} \qquad \text{and} \qquad \sigma_p = \boldsymbol{P} \cdot \boldsymbol{n},$$

where $\boldsymbol{n}$ is a unit vector pointing out of the dielectric.

(b) Hence define the displacement $\boldsymbol{D}$ and show that

$$\nabla \cdot \boldsymbol{D} = \rho,$$

where ρ is the volume density of free charge.

(c) A cable consists of concentric conducting cylinders of radii a and b, $a < b$. The space between the cylinders is filled by a dielectric of constant κ. Given that the inner conductor carries a charge λ per unit length, determine the polarization charge densities ρ_p and σ_p within the cable. Neglect edge effects.

(d) Find the capacity per unit length of the cable.

[Exam 2004 Question 2]

2. (a) Using Gauss's law, show that the electric field of a uniformly charged sphere with charge density ρ and radius R is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho}{3\varepsilon_0}\boldsymbol{r} \quad \text{(inside)}, \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho R^3}{3\varepsilon_0}\frac{\boldsymbol{r}}{r^3} \quad \text{(outside)}.$$

(b) Consider two spheres of radius R, uniformly charged with charge densities $\rho > 0$ and $-\rho$. Let the centre of the positively charged sphere is displaced from the centre of the negatively charged sphere by vector $\boldsymbol{d}$, so that the spheres partially overlap.
Using the superposition principle and the result from part (a), show that in the space where the two spheres overlap, the electric field is uniform and given by

$$\boldsymbol{E} = -\frac{\rho \boldsymbol{d}}{3\varepsilon_0}.$$

(c) From now on assume that $d \ll R$. Show that in this case the total charge distribution of the two spheres is equivalent to that of a sphere with the *surface* charge density

$$\sigma_p = \rho d \cos\theta,$$

where θ is the angle between $\boldsymbol{d}$ and the direction to the point on the sphere.
[Hint: consider the thickness of the thin layer where the positively and negatively charged spheres do not overlap.]

(d) If the positive and negative charge distributions from part (b) and (c) are due to *bound*, polarisation, charges, the corresponding polarisation vector is $\boldsymbol{P} = \rho \boldsymbol{d}$.
Hence, verify that $\rho_p = 0$ inside the sphere and $\boldsymbol{P} \cdot \boldsymbol{n} = \sigma_p$ on its surface, where $\boldsymbol{n}$ is the outer normal.

(e) Show that the electric field outside the sphere is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5},$$

where $\boldsymbol{p} = \frac{4}{3}\pi R^3 \rho \boldsymbol{d}$ is the total dipole moment of the sphere (equal to $\boldsymbol{P}V$, V being the volume of the sphere).
[Hint: consider the superposition of the fields *outside* of the positively and negatively uniformly charged spheres whose centres are separated by $\boldsymbol{d}$, such that $d \ll R$.]