Electromagnetic Theory AMA3001 Problem sheet 3

Method of images. Solutions of Laplace's equation.

Homework

1. (a) An electrostatic system consists of a set of conductors, each carrying a given total charge; another set of conductors, each kept at a given potential; and a given volume distribution of charge. No dielectrics are present.

   Show that the electrostatic potential for this system is unique.

   (b) A point dipole is placed at a distance a from an infinite conducting plane held at zero potential. If the the z-axis is normal to the plane then the position of the dipole is $a\boldsymbol{k}$ and the dipole moment is $p\boldsymbol{k}$.

   i. Using the method of images find the electrostatic potential $\phi(x, y, z)$ when $z > 0$.

   ii. Find the surface charge density $\sigma(x, y)$ on the conductor.

   [Exam 2008 Question 2]

2. (a) A point charge q is distance b from a point O. Show that it is possible to obtain zero potential on a spherical surface of radius a, where $a < b$, centred on O by placing a second point charge on the line Oq. What is the magnitude and position of this second charge?

   (b) A point dipole $\boldsymbol{p}$ is at position $(0, 0, d)$ on the z-axis of a Cartesian coordinate system with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. A conducting sphere of radius a, with $a < d$, and at zero potential is centred upon the origin.

   Treat the point dipole by considering the image of a dipole and taking the point-dipole limit.

   i. If $\boldsymbol{p} = p\boldsymbol{k}$, show that the electrostatic image of $\boldsymbol{p}$ in the sphere is a dipole of moment $(pa^3/d^3)\boldsymbol{k}$ and a point charge of magnitude pa/d^2, both at $(0, 0, a^2/d)$.

   ii. If $\boldsymbol{p} = p\boldsymbol{i}$, show that the image is a dipole of moment $-(pa^3/d^3)\boldsymbol{i}$.

   iii. Deduce the image system for $\boldsymbol{p} = p\sin\alpha\,\boldsymbol{i} + p\cos\alpha\,\boldsymbol{k}$ (using the principle of superposition).

3. (a) A line charge with linear charge density λ is placed at a distance a from an infinite conducting plane at zero potential. Show that at a point P on the plane the density of induced charge is $-\lambda a/\pi r^2$, where r is the shortest distance from P to the line charge.

   (b) A line charge with linear charge density λ is parallel to, and a distance b away from, the axis of an infinite conducting cylinder of radius a $(a < b)$. Find the position of the image line charge inside the cylinder, assuming that the total charge per unit length of the whole system is zero, and show that in cylindrical polar coordinates (ρ, ψ, z) the electrostatic potential outside the cylinder can be written as

   $$\phi(\rho, \psi) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\rho^2 + b^2 - 2\rho b\cos\psi}{(\rho b/a)^2 + a^2 - 2\rho b\cos\psi}\right].$$

   (c) What additional image charge needs to be placed, and where, for the cylinder to carry no net charge?

   [The electrostatic potential of a uniformly charged line is $\phi = -\dfrac{\lambda}{2\pi\varepsilon_0}\ln\rho + \text{const}$, where ρ is the distance from the line (see homework question 5 from Problem sheet 1).]

4. (a) Consider a boundary surface between two dielectric materials, labelled 1 and 2. If $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ are the electric fields in the two materials, and $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ are the corresponding electric displacements, then prove the following boundary conditions

$$\begin{aligned} E_{1t} &= E_{2t}, \\ D_{1n} &= D_{2n} + \sigma, \end{aligned}$$

where subscript t designates the component tangential to the surface and subscript n designates the component normal to the surface. The surface charge density is σ.

How are the boundary conditions changed when material 2 is a conductor?

(b) A conducting sphere of radius a is placed in vacuum in a uniform electric field $\boldsymbol{E}_0$. The total charge on the conducting sphere is Q.

Determine the potential everywhere and the surface charge density on the conductor.

[Exam 2004 Question 4]

5. A dielectric sphere of radius a and uniform dielectric constant κ is placed in vacuum in a uniform electric field $\boldsymbol{E}_0$. Assuming that the electrostatic potential takes the general form

$$\phi(r, \theta) = \frac{A}{r^2} \cos\theta + \frac{B}{r} + C + Dr\cos\theta,$$

where A, B, C, D are constants to be determined, and (r, θ) are spherical polar coordinates, with the z axis in the direction of $\boldsymbol{E}_0$, find the potential both outside and inside the sphere.

Show that the volume polarization charge density is zero, and determine the surface polarization charge density.

[Exam 2005 Question 3, problem part; bookwork part was the same as Question 4 above]

6. An uncharged conducting cylinder of radius a and infinite length is placed with its axis perpendicular to a uniform field $\boldsymbol{E}_0$. Calculate the potential at all points and deduce that the greatest surface density of induced charge is $2\varepsilon_0 E_0$.

[Hint: the potential does not depend on z (the coordinate along the cylinder), so we can use the general solution of Laplace's equation in cylindrical coordinates in the form

$$\phi = C \ln\rho + \sum_{n=-\infty}^{\infty} (A_n \cos n\psi + B_n \sin n\psi)\rho^n,$$

where C, A_n and B_n are the coefficients determined from boundary and other conditions.]

Further Examples

1. (a) Consider a finite electrostatic system consisting of a set of conductors each carrying a given total charge, another set of conductors maintained at prescribed potentials, and a volume distribution of charge.

   Show that the electrostatic potential for this system is unique.

   (b) A point charge q is placed in vacuum outside an infinite conducting plane which is held at zero potential. The perpendicular distance from the charge to the plane is a.

   i. Find the electrostatic potential in the half-plane containing the charge q.

   ii. Show that the charge density σ on the surface of the conductor is given by

   $$\sigma(\rho) = -\frac{aq}{2\pi(a^2+\rho^2)^{3/2}},$$

   where ρ is the distance measured on the plane from the point where the perpendicular from the charge q meets the plane.

   iii. Verify that the total charge on the plane is $-q$.

   iv. Determine the magnitude and direction of the force experienced by the charge q.

   [Exam 2005 Question 2]

2. A point dipole of moment $\boldsymbol{p}$ is at the point $(0, 0, d)$ on the z-axis of a Cartesian coordinate system. The moment vp makes an angle θ with the z-axis and is oriented so that

$$\boldsymbol{p} = p\sin\theta\,\boldsymbol{j} + p\cos\theta\,\boldsymbol{k}\,.$$

The x-y plane is an infinite conducting plane held at zero potential.

   (a) Show that the potential at a point with coordinates (x, y, z) is

$$\phi(x,y,z) = \frac{p}{4\pi\varepsilon_0}\left\{\frac{y\sin\theta+(z-d)\cos\theta}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{-y\sin\theta+(z+d)\cos\theta}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}.$$

   (b) Show that the surface charge density on the conducting plane is

$$\sigma(x,y) = -\frac{p}{2\pi(x^2+y^2+d^2)^{3/2}}\left[\cos\theta + 3d\left(\frac{y\sin\theta - d\cos\theta}{x^2+y^2+d^2}\right)\right],$$

   and verify by direct integration that the total induced charge is zero.

3. A point charge q is placed in a vacuum outside a conducting sphere of radius a, at a distance b from the centre of the sphere which is maintained at zero potential.

   (a) Explain why the same potential in the region $r > a$ is obtained if the sphere is replaced by an image point charge $-aq/b$ at a distance a^2/b from the centre of the sphere and determine this potential.

   (b) Determine the surface charge density σ on the sphere.

   (c) Determine the force on the point charge q.

   [Exam 2003 Question 2, problem part; bookwork part was the same as in Question 1 above]

4. A point dipole $\boldsymbol{p}$ is placed in a vacuum outside a conducting sphere of radius a. The dipole is at a distance d from the centre of the sphere which is maintained at zero potential. The direction of the dipole is radially away from the centre of the sphere.

   (a) Find the image system by considering a dipole and then taking the point dipole limit.

   (b) Write down the potential for $r > a$.

   (c) Verify that the potential is zero when $r = a$.

   (d) Explain why the total charge on the conductor is non-zero.

   You may assume that for a point charge q at a distance b, the image is a charge $-aq/b$ at a distance a^2/b.

   [Exam 2004 Question 3, problem part; bookwork part was the same as in Question 1 above]

5. A dielectric sphere of radius a and permittivity ε has its centre at the origin and is uniformly charged with charge density ρ_0. It is surrounded concentrically with a conducting shell occupying the region $a < r < b$ and the whole system is placed in a vacuum in a uniform electric field $\boldsymbol{E}_0$. If the total charge on the conducting sphere is zero, show that, in spherical polar coordinates (r, θ, ψ), the potential is given by

$$\phi(r,\theta) = \begin{cases} \dfrac{\rho_0 a^2}{6}\left[\dfrac{1}{\varepsilon}\left(1 - \dfrac{r^2}{a^2}\right) + \dfrac{2a}{\varepsilon_0 b}\right], & r < a \text{ (inside dielectric)} \\ \dfrac{\rho_0 a^3}{3\varepsilon_0 b}, & a < r < b \text{ (inside conducting shell)} \\ -E_0 r\cos\theta\left(1 - \dfrac{b^3}{r^3}\right) + \dfrac{\rho_0 a^3}{3\varepsilon_0 r}, & r > b \text{ (outside conducting shell)} \end{cases}$$

6. An uncharged conducting sphere of radius a is inside a concentric dielectric shell of dielectric constant κ which fills the space $a < r \leq b$. They are placed with their centre at the origin in a uniform electric field $\boldsymbol{E}_0$. Show that, in spherical polar coordinates (r, θ, ψ), the potential is given by

$$\phi(r,\theta) = \begin{cases} -\dfrac{3E_0 b^3}{[(\kappa+2)b^3 + 2(\kappa-1)a^3]}\left(1 - \dfrac{a^3}{r^3}\right) r\cos\theta, & a < r < b \text{ (inside dielectric)} \\ -E_0 r\cos\theta + \dfrac{E_0 b^3[(\kappa-1)b^3 + (2\kappa+1)a^3]}{[(\kappa+2)b^3 + 2(\kappa-1)a^3]}\dfrac{\cos\theta}{r^2}, & r > b \text{ (outside dielectric)} \end{cases}$$

   Show also that the surface charge density induced on the spherical conductor is

$$\sigma(\theta) = \frac{9\varepsilon_0 \kappa E_0 b^3}{[(\kappa+2)b^3 + 2(\kappa-1)a^3]}\cos\theta.$$