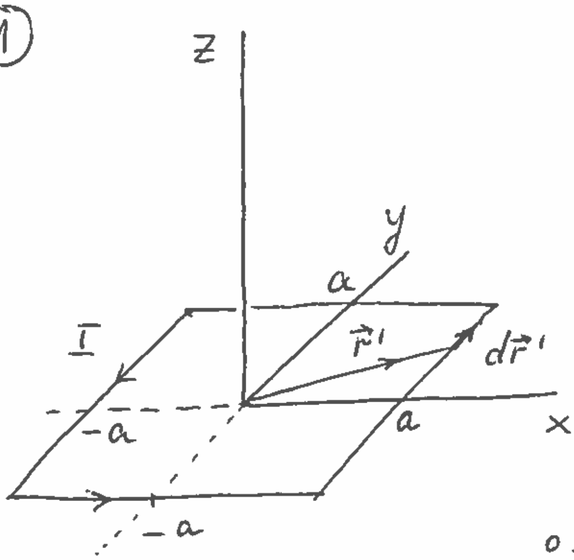
AMA3001 Electromagnetic Theory

Problem sheet 5

SOLUTIONS

(1)



The magnetic field produced by circuit C carrying current I is

$$B(\vec{r}) = \frac{\mu_0}{4\pi} I \int_C \frac{d\vec{r}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

In our case C is the square of side 2a in the x-y plane, and we need to find the field at the origin, $\vec{r} = 0$.

Hence, $\quad d\vec{r}' \times (\vec{r} - \vec{r}') = -d\vec{r}' \times \vec{r}' = \vec{r}' \times d\vec{r}'$.

Both $\vec{r}'$ and $d\vec{r}'$ lie in the x-y plane, which means that $\vec{r}' \times d\vec{r}'$ is perpendicular to it, along the z axis. By symmetry, the contributions of all four sides of the square are equal. Therefore,

$$\vec{B} = \frac{\mu_0}{4\pi} I\, 4 \int_{\substack{\text{one}\\ \text{side}}} \frac{\vec{r}' \times d\vec{r}'}{|\vec{r}'|^3} \qquad (*)$$

Let us use the side parallel to the y axis which crosses the x axis at $x = a$. For this side

$$\vec{r}' = a\vec{i} + y\vec{j},$$

$$d\vec{r}' = dy\,\vec{j}$$

$$\vec{r}' \times d\vec{r}' = (a\vec{i} + y\vec{j}) \times \vec{j}\, dy = a\,dy\, \vec{i} \times \vec{j} = a\,dy\,\vec{k}.$$

$$|\vec{r}'| = \sqrt{a^2 + y^2}.$$

Substituting these into (*), we have:

$$\vec{B} = \frac{\mu_0}{\pi} I \int_{-a}^{a} \frac{a\,dy}{\left(\sqrt{a^2+y^2}\right)^3}\,\vec{k} = \frac{\mu_0 I a}{\pi}\int_{-a}^{a}\frac{dy}{(a^2+y^2)^{3/2}}\,\vec{k}$$

Let's use variable substitution

$$y = a\tan\alpha, \qquad dy = \frac{a}{\cos^2\alpha}\,d\alpha$$

$$a^2+y^2 = a^2(1+\tan^2\alpha) = \frac{a^2}{\cos^2\alpha}.$$

Limits:

$y = a \Leftrightarrow \alpha = \frac{\pi}{4}$

$y = -a \Leftrightarrow \alpha = -\frac{\pi}{2}$

Hence:

$$\vec{B} = \frac{\mu_0 I a}{\pi}\,\vec{k}\int_{-\pi/4}^{\pi/4}\frac{a\,d\alpha}{\cos^2\alpha}\,\frac{\cos^3\alpha}{a^3}$$

$$= \frac{\mu_0 I}{\pi a}\,\vec{k}\int_{-\pi/4}^{\pi/4}\cos\alpha\,d\alpha$$

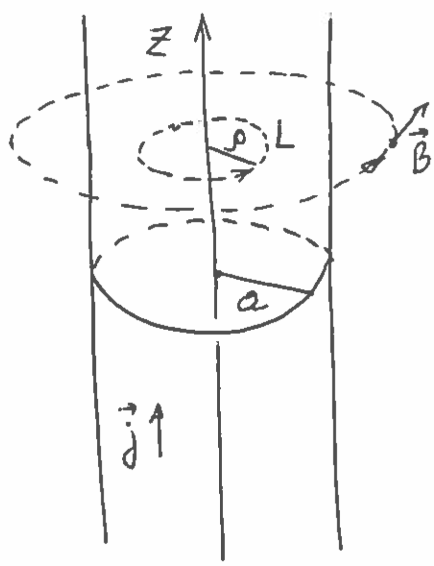
$$= \frac{\mu_0 I}{\pi a}\,\vec{k}\,\big[\sin\alpha\big]_{-\pi/4}^{\pi/4}$$

$$= \frac{\mu_0 I}{\pi a}\,\vec{k}\left[\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\mu_0 I}{\pi a}\,\frac{2}{\sqrt{2}}\,\vec{k}$$

$$\Rightarrow \quad \vec{B} = \frac{\sqrt{2}\,\mu_0 I}{\pi a}\,\vec{k}, \quad \text{as required.}$$

2 (a)



By the symmetry of the system, the magnetic field depends only on the distance ρ from the z axis. Since the lines of $\vec{B}$ are closed, they are circles centred on the z axis, and the magnetic field is tangential to such circles. We find it by using

$$\oint_L \vec{B}\cdot d\vec{r} = \mu_0 I_L, \qquad (*)$$

where L is a circle of radius ρ (either $\rho \le a$, or $\rho > a$) and I_L is the total current through such circle. Since the current is uniform across the wire, the current density $\vec{j}$ (along the z axis) has the magnitude

$$j = \frac{I}{\pi a^2}.$$

Applying (*) to $\rho \le a$ first, we take into account that $\vec{B}$ is parallel to $d\vec{r}$ at every point, so that

$$\oint \vec{B} \cdot d\vec{r} = \oint B(\rho)\, dr = B(\rho) \oint dr = B(\rho) 2\pi\rho .$$

Therefore (*) gives

$$B\, 2\pi\rho = \mu_0 j \pi\rho^2 \qquad (\rho \le a)$$

$$B = \frac{\mu_0 j \pi\rho^2}{2\pi\rho} = \frac{\mu_0 j \rho}{2} = \frac{\mu_0 I \rho}{2\pi a^2}.$$

The direction of $\vec{B}$ is perpendicular to the z axis and to ρ, in the direction of the unit vector $\hat{\varphi}$:

$$\vec{B} = \frac{\mu_0 I \rho}{2\pi a^2} \hat{\varphi}$$

For $\rho > a$:

$$B\, 2\pi\rho = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi\rho}$$

and

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\varphi}.$$

(b) To find the vector potential $\vec{A}$, we use $\vec{B} = \vec{\nabla} \times \vec{A}$ and the expression for the curl operator in cylindrical coordinates:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$$

, and given that $\vec{A} = A_z(\rho)\vec{k}$,

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = \frac{1}{\rho}\hat{\rho}\underbrace{\frac{\partial A_z}{\partial \varphi}}_{=0} - \hat{\varphi}\frac{\partial A_z}{\partial \rho} = -\hat{\varphi}\frac{dA_z}{d\rho},$$

since A_z depends on ρ only.

For $\rho \leq a$:

$$-\hat{\varphi}\frac{A_z}{d\rho} = \frac{\mu_0 I \rho}{2\pi a^2}\hat{\varphi} \qquad \text{(from part (a))}$$

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I \rho}{2\pi a^2}$$

$$A_z = -\int \frac{\mu_0 I \rho}{2\pi a^2}\, d\rho$$

$$A_z = -\frac{\mu_0 I \rho^2}{4\pi a^2} + C,$$

where we can set the arbitrary constant $C = 0$.

Then

$$A_z(\rho) = -\frac{\mu_0 I \rho^2}{4\pi a^2}$$

For $\rho > a$

$$-\frac{dA_z}{d\rho} = \frac{\mu_0 I}{2\pi\rho}$$

$$A_z = -\int \frac{\mu_0 I}{2\pi\rho}\, d\rho$$

$$A_z = -\frac{\mu_0 I}{2\pi}\ln\rho + D$$

arbitrary constant.

The constant D must be chosen so that the vector potential for $\rho > a$ is equal to that for $\rho \leq a$ at the boundary $\rho = a$. This gives:

$$-\frac{\mu_0 I}{2\pi}\ln a + D = -\frac{\mu_0 I a^2}{4\pi a^2}$$

$$\Rightarrow \quad D = \frac{\mu_0 I}{2\pi}\ln a - \frac{\mu_0 I}{4\pi}.$$

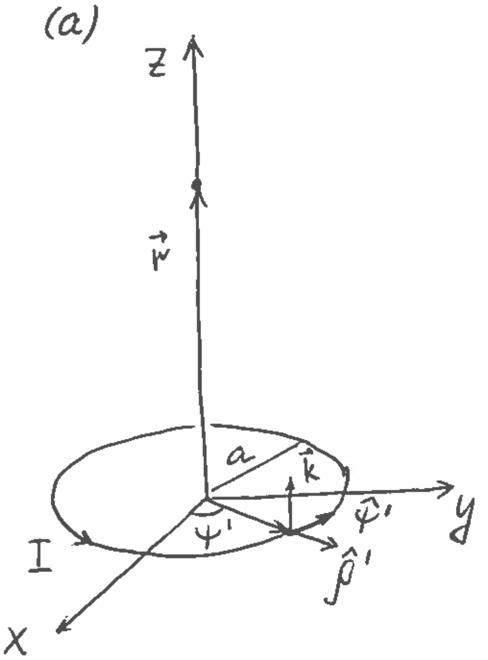
Hence, for $\rho > a$, we have:

$$A_z(\rho) = -\frac{\mu_0 I}{2\pi}\ln\rho + \frac{\mu_0 I}{2\pi}\ln a - \frac{\mu_0 I}{4\pi}$$

$$= -\frac{\mu_0 I}{4\pi}\left(1 + 2\ln\rho - 2\ln a\right)$$

$$= -\frac{\mu_0 I}{4\pi}\left(1 + 2\ln\frac{\rho}{a}\right).$$

3 (a)



The magnetic field is found using

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int_C \frac{d\vec{r}\,' \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

On the z axis,

$$\vec{r} = z\vec{k}$$

and for the point on the ring,

$$\vec{r}\,' = a\hat{\rho}'$$

$$d\vec{r}\,' = a\,d\varphi'\,\hat{\varphi}'$$

[$\hat{\rho}' = \cos\varphi'\,\vec{i} + \sin\varphi'\,\vec{j}$,

$d\hat{\rho}' = (-\sin\varphi'\,\vec{i} + \cos\varphi'\,\vec{j})\,d\varphi' = \hat{\varphi}'\,d\varphi'$,

where $\hat{\varphi}' = -\sin\varphi'\,\vec{i} + \cos\varphi'\,\vec{j}$ is the unit vector in φ direction.]

$$d\vec{r}\,' \times (\vec{r}-\vec{r}\,') = a\,d\varphi'\,\hat{\varphi}' \times (z\vec{k} - a\hat{\rho}')$$

$$= a\,d\varphi'\big(z\underbrace{\hat{\varphi}'\times\vec{k}}_{=\hat{\rho}'} - a\underbrace{\hat{\varphi}'\times\hat{\rho}'}_{=-\vec{k}}\big) = a\,d\varphi'(z\hat{\rho}' + a\vec{k})$$

$$|\vec{r}-\vec{r}\,'| = |z\vec{k} - a\hat{\rho}'| = \sqrt{a^2+z^2}$$

Therefore,

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a\,d\varphi'\,(z\hat{\rho}' + a\vec{k})}{(a^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi} \frac{a}{(a^2+z^2)^{3/2}} \left[z \int_0^{2\pi} \hat{\rho}\, d\varphi' + a\hat{k} \int_0^{2\pi} d\varphi' \right]$$

The first integral gives zero:

$$\int_0^{2\pi} (\cos\varphi' \vec{i} + \sin\varphi' \vec{j})\, d\varphi' = \vec{i} \int_0^{2\pi} \cos\varphi' d\varphi' + \vec{j} \int_0^{2\pi} \sin\varphi' d\varphi' = 0$$

The 2nd integral gives:

$$\int_0^{2\pi} d\varphi' = 2\pi.$$

Hence, we obtain:

$$\vec{B} = \frac{\mu_0 I\, a^2\, 2\pi}{4\pi\,(a^2+z^2)^{3/2}}\, \vec{k},$$

or

$$\underline{\underline{\vec{B} = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}\, \vec{k}}},$$ as required.

(b) The vector potential of the current is found using the formula

$$A(\vec{r}) = \frac{\mu_0}{4\pi} I \int_C \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|},$$

where

$$\vec{r} = \rho\hat{\rho} + z\vec{k}$$

$$= \rho(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) + z\vec{k}$$

Here $\vec{r}$ is an arbitrary point

$$\vec{r}\,' = a\hat{\rho}' = a(\cos\varphi'\,\vec{i} + \sin\varphi'\,\vec{j})$$

$$d\vec{r}' = a(-\sin\varphi'\,\vec{i} + \cos\varphi'\,\vec{j})\,d\varphi'$$

$$|\vec{r}-\vec{r}'| = \left[(\rho\cos\varphi - a\cos\varphi')^2 + (\rho\sin\varphi - a\sin\varphi')^2 + z^2\right]^{1/2}$$

$$= \left\{\rho^2(\cos^2\varphi + \sin^2\varphi) + a^2(\cos^2\varphi' + \sin^2\varphi') - 2a\rho\left[\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\right] + z^2\right\}^{1/2}$$

$$= \sqrt{\rho^2 + a^2 + z^2 - 2a\rho\cos(\varphi'-\varphi)}$$

Therefore,

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{(-a\sin\varphi'\,\vec{i} + a\cos\varphi'\,\vec{j})\,d\varphi'}{\sqrt{\rho^2+a^2+z^2-2a\rho\cos(\varphi'-\varphi)}}.$$

(c) Let $\varphi'-\varphi = \beta$, $\varphi' = \varphi+\beta$, $d\varphi' = d\beta$

$$-a\sin\varphi'\,\vec{i} + a\cos\varphi'\,\vec{j} = -a\sin(\varphi+\beta)\,\vec{i} + a\cos(\varphi+\beta)\,\vec{j}$$

$$= -a(\sin\varphi\cos\beta + \cos\varphi\sin\beta)\,\vec{i} + a(\cos\varphi\cos\beta - \sin\varphi\sin\beta)\,\vec{j}$$

Hence, we have:

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-a(\sin\varphi\cos\beta + \cos\varphi\sin\beta)\,\vec{i} + a(\cos\varphi\cos\beta - \sin\varphi\sin\beta)\,\vec{j}}{\sqrt{\rho^2+a^2+z^2-2\rho a\cos\beta}}\,d\beta$$

Note that the integrand is a 2π-periodic function of β, and since the integral is over <u>one period</u>, the limits can be chosen as $-\varphi$ and $2\pi-\varphi$ (as from $\beta = \varphi'-\varphi$), or 0 to 2π, or <u>$-\pi$ to π</u>. In the latter case the integration interval is symmetric and only even functions of β give a nonzero contribution. The denominator is even and terms with $\cos\beta$ in the numerator are even, but the terms with $\sin\beta$ are odd $\Rightarrow$ they do not contribute. Therefore:

$$A(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-\pi}^{\pi} \frac{-a\sin\varphi\cos\beta\,\vec{i} + a\cos\varphi\cos\beta\,\vec{j}}{\sqrt{\rho^2+a^2+z^2-2a\rho\cos\beta}}\,d\beta$$

$$= \frac{\mu_0 I a}{4\pi}\int_{-\pi}^{\pi} \frac{\cos\beta\,d\beta}{\sqrt{\rho^2+a^2+z^2-2a\rho\cos\beta}}\underbrace{(-\sin\varphi\,\vec{i}+\cos\varphi\,\vec{j})}_{\hat{\varphi}\text{ unit vector}}$$

$$= \frac{\mu_0 I a}{2\pi}\int_{0}^{\pi} \frac{\cos\beta\,d\beta}{\sqrt{\rho^2+a^2+z^2-2a\rho\cos\beta}}\,\hat{\varphi}$$

[In the last step we used the fact that $\int_{-\pi}^{\pi}\ldots = 2\int_{0}^{\pi}\ldots$ for an even function.]

Looking at the expected answer, we see $\sin^2\alpha$ under the square roots in the elliptic integrals, and the limits being 0 and $\pi/2$. Hence we use the double angle formula and introduce $2\alpha = \beta$, $d\beta = 2\,d\alpha$.

$$\cos\beta = \cos 2\alpha = 2\cos^2\alpha - 1$$

$$\rho^2+a^2+z^2-2a\rho\cos\beta = \rho^2+a^2+z^2-2a\rho(2\cos^2\alpha-1)$$

$$= \rho^2+2a\rho+a^2+z^2-4a\rho\cos^2\alpha$$

$$= (a+\rho)^2+z^2-4a\rho\cos^2\alpha$$

$$= \left[(a+\rho)^2+z^2\right]\left[1-\frac{4a\rho}{(a+\rho)^2+z^2}\cos^2\alpha\right]$$

$$= \left[(a+\rho)^2+z^2\right](1-k^2\cos^2\alpha),$$

where $$k^2 = \frac{4a\rho}{(a+\rho)^2+z^2}.$$

Hence, the integral on page 8 is transformed as follows:

$$A(\vec{r}) = \frac{\mu_0 I a}{2\pi}\hat{\varphi}\int_0^{\pi/2}\frac{(2\cos^2\alpha - 1)\,2\,d\alpha}{\sqrt{(a+\rho)^2+z^2}\sqrt{1-k^2\cos^2\alpha}}$$

$$= \frac{\mu_0 I a\,\hat{\varphi}}{\pi\sqrt{(a+\rho)^2+z^2}}\int_0^{\pi/2}\frac{(2\cos^2\alpha-1)\,d\alpha}{\sqrt{1-k^2\cos^2\alpha}}$$

$$= \frac{\mu_0 I a\,\hat{\varphi}}{\pi\sqrt{(a+\rho)^2+z^2}}\,\frac{2}{k^2}\int_0^{\pi/2}\frac{\left(k^2\cos^2\alpha - \frac{k^2}{2}\right)d\alpha}{\sqrt{1-k^2\cos^2\alpha}}$$

$$= \frac{\mu_0 I a\,\hat{\varphi}}{\pi\sqrt{(a+\rho)^2+z^2}}\,\frac{\sqrt{4a\rho}}{\sqrt{a\rho}\,k^2}\int_0^{\pi/2}\frac{-\frac{k^2}{2}+1-(1-k^2\cos^2\alpha)}{\sqrt{1-k^2\cos^2\alpha}}\,d\alpha$$

Here we are working towards the answer

$$= \frac{\mu_0 I\,\hat{\varphi}}{\pi k}\sqrt{\frac{a}{\rho}}\left[\left(1-\frac{k^2}{2}\right)\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\cos^2\alpha}} - \int_0^{\pi/2}\sqrt{1-k^2\cos^2\alpha}\,d\alpha\right]$$

In the remaining integrals we can replace $\alpha \to \frac{\pi}{2}-\alpha$.

After this $\cos^2\alpha \to \cos^2\left(\frac{\pi}{2}-\alpha\right) = \sin^2\alpha$

$d\alpha \to -d\alpha$

Limits $\int_0^{\pi/2}\dots \to \int_{\pi/2}^{0}\dots = -\int_0^{\pi/2}\dots$

Hence we obtain:

$$A(\vec{r}) = \frac{\mu_0 I}{\pi k}\sqrt{\frac{a}{\rho}}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right],$$

where $$K(k) = \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}},$$

and $$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,d\alpha,$$ as required.

(d) Near the axis of the loop ρ is small and the parameter

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2} \ll 1.$$

Hence, we can use binomial expansions

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{(-1/2)(-3/2)}{2}x^2 + \dots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \dots ,$$

and $$(1+x)^{1/2} = 1 + \tfrac{1}{2}x + \frac{1/2\,(-1/2)}{2}x^2 + \dots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \dots ,$$

to evaluate the elliptic integrals $K(k)$ and $E(k)$. Note that we need to expand both to second order, since the zeroth- and first-order contributions will cancel (see below).

Therefore,

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\alpha)^{-1/2}\, d\alpha$$

$$\simeq \int_0^{\pi/2} \left[1 + \tfrac{1}{2}k^2 \sin^2\alpha + \tfrac{3}{8}k^4 \sin^4\alpha\right] d\alpha$$

$$= \frac{\pi}{2} + \frac{1}{2}k^2 \int_0^{\pi/2} \sin^2\alpha\, d\alpha + \frac{3}{8}k^4 \int_0^{\pi/2} \sin^4\alpha\, d\alpha$$

$$\int_0^{\pi/2} \sin^2\alpha\, d\alpha = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2\alpha)\, d\alpha = \frac{1}{2}\left[\int_0^{\pi/2} d\alpha - \int_0^{\pi/2} \cos 2\alpha\, d\alpha\right]$$

$[\cos 2\alpha = 1 - 2\sin^2\alpha]$ $$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\left[\sin 2\alpha\right]_0^{\pi/2}\right] = \frac{\pi}{4}.$$

$$\int_0^{\pi/2} \sin^4\alpha\, d\alpha = \frac{1}{4}\int_0^{\pi/2} (1 - \cos 2\alpha)^2\, d\alpha = \frac{1}{4}\int_0^{\pi/2} (1 - 2\cos 2\alpha + \cos^2 2\alpha)\, d\alpha$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - 0 + \frac{1}{2}\int_0^{\pi/2}(1+\cos 4\alpha)\,d\alpha\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} + 0\right] = \frac{1}{4}\ \frac{3\pi}{4} = \frac{3\pi}{16}$$

Hence,

$$K(k) \simeq \frac{\pi}{2} + \frac{\pi}{8}k^2 + \frac{3}{8}\cdot\frac{3\pi}{16}k^4 = \frac{\pi}{2} + \frac{\pi}{8}k^2 + \frac{9\pi}{128}k^4 .$$

Similarly,

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2\alpha)^{1/2}\,d\alpha$$

$$\simeq \int_0^{\pi/2}\left[1 - \frac{1}{2}k^2\sin^2\alpha - \frac{1}{8}k^4\sin^4\alpha\right]d\alpha$$

$$= \frac{\pi}{2} - \frac{1}{2}k^2\frac{\pi}{4} - \frac{1}{8}k^4\cdot\frac{3\pi}{16}$$

$$= \frac{\pi}{2} - \frac{\pi}{8}k^2 - \frac{3\pi}{128}k^4 .$$

Hence, the vector potential near the axis is given by:

$$A(\vec{r}) \simeq \frac{\mu_0 I}{\pi k}\sqrt{\frac{a}{\rho}}\left[\left(1-\frac{k^2}{2}\right)\left(\frac{\pi}{2} + \frac{\pi}{8}k^2 + \frac{9\pi}{128}k^4\right) - \left(\frac{\pi}{2} - \frac{\pi}{8}k^2 - \frac{3\pi}{128}k^4\right)\right]\hat{\varphi}$$

$$\simeq \frac{\mu_0 I}{\pi k}\sqrt{\frac{a}{\rho}}\left[\cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{8}k^2} + \frac{9\pi}{128}k^4 - \cancel{\frac{\pi}{4}k^2} - \frac{\pi}{16}k^4 - \cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{8}k^2} + \frac{3\pi}{128}k^4 + O(k^6)\right]\hat{\varphi}$$

(we only keep terms up to k^4).

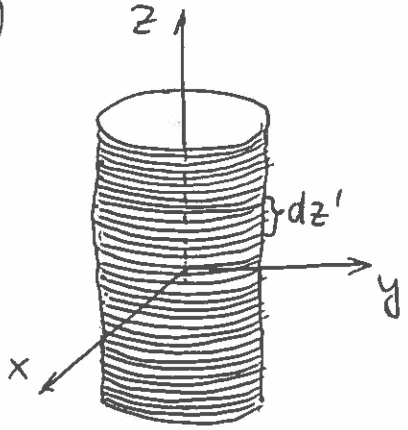
$$= \frac{\mu_0 I}{\pi k}\sqrt{\frac{a}{\rho}}\ \frac{\pi}{16}\left(\frac{9}{8} - 1 + \frac{3}{8}\right)k^4\,\hat{\varphi}$$

$$= \frac{\mu_0 I}{\pi}\sqrt{\frac{a}{\rho}}\ \frac{\pi}{16}\cdot\frac{1}{2}k^3\hat{\varphi} = \frac{\mu_0 I}{32}\sqrt{\frac{a}{\rho}}\left[\frac{4a\rho}{(a+\rho)^2+z^2}\right]^{3/2}\hat{\varphi},$$

which finally gives:

$$A(\vec{r}) = \frac{\mu_0 I a^2}{4} \frac{\rho}{[(a+\rho)^2 + z^2]^{3/2}} \hat{\varphi} .$$

4



The solenoid can be considered as a large number (N) of circular loops, such as that considered in question 3.

The magnetic field on the axis of a circular loop is

$$\vec{B} = \frac{\mu_0 I a^2}{2(a^2 + (z-z'))^{3/2}} \vec{k} ,$$

where z is the point at which we observe B and z' is the z coordinate of the loop.

The number of loops per interval dz' is

$$dN = \frac{dz'}{L} N ,$$

where L is the length of the solenoid.

Hence the total magnetic field (in the z direction) is given by:

$$B = \frac{\mu_0 I a^2}{2} \int_{-L/2}^{L/2} \frac{1}{[a^2 + (z-z')^2]^{3/2}} \frac{dz'}{L} N$$

$$= \frac{\mu_0 N I a^2}{2L} \int_{-L/2}^{L/2} \frac{dz'}{[a^2 + (z'-z)^2]^{3/2}} .$$

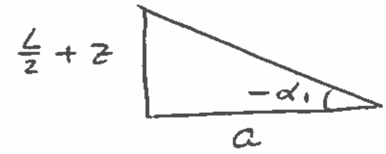
Using variable substitution,

$$z' - z = a \tan\alpha ,$$

$$dz' = \frac{a}{\cos^2\alpha}\,d\alpha$$

Integration limits:

$$-\frac{L}{2} - z = a\tan\alpha_1$$



$$\frac{L}{2} - z = a\tan\alpha_2$$

Hence, we have:

$$B = \frac{\mu_0 N I a^2}{2L}\int_{\alpha_1}^{\alpha_2} \frac{a\,d\alpha}{\cos^2\alpha}\,\frac{\cos^3\alpha}{a^3}$$

$$\left\{ \begin{aligned} &a^2 + (z'-z)^2 \\ &= a^2(1+\tan^2\alpha) \\ &= \frac{a^2}{\cos^2\alpha} \end{aligned} \right.$$

$$= \frac{\mu_0 N I}{2L}\int_{\alpha_1}^{\alpha_2} \cos\alpha\,d\alpha$$

$$= \frac{\mu_0 N I}{2L}\Big[\sin\alpha\Big]_{\alpha_1}^{\alpha_2}$$

$$= \frac{\mu_0 N I}{2L}\Big[\sin\alpha_2 - \sin\alpha_1\Big]$$

Here we use the above diagrams of triangles.

$$= \frac{\mu_0 N I}{2L}\left[\frac{L/2 - z}{\sqrt{(L/2-z)^2 + a^2}} + \frac{L/2 + z}{\sqrt{(L/2+z)^2 + a^2}}\right],$$

as required.

(5) (a) For a steady volume distribution of current with current density $\vec{j}$, the magnetic induction is given by the integral over the volume containing the currents.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV'.$$

(b) $$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}\cdot\left[\frac{\vec{j}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right] dV',$$

since the integral is with respect to $\vec{r}\,'$ and $\vec{\nabla}$ acts on $\vec{r}$.

Using the identity $\quad \vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} = - \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$,

we have:

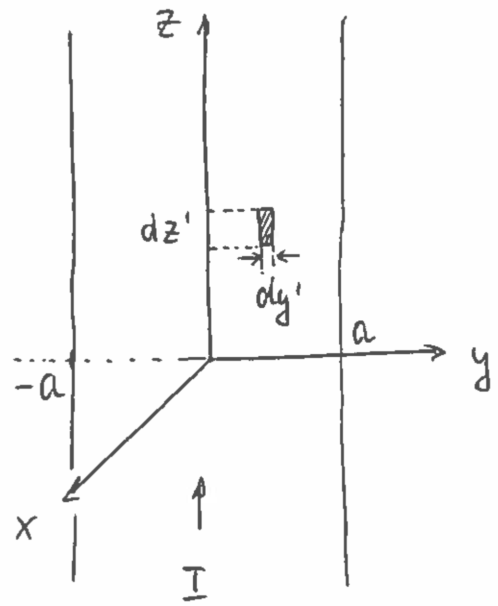
$$\vec{\nabla}\cdot\vec{B} = -\frac{\mu_0}{4\pi}\int_V \vec{\nabla}\cdot\left(\vec{j}(\vec{r}\,')\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dV'$$

Using $\quad \vec{\nabla}\cdot(\vec{a}\times\vec{\nabla}f) = -\vec{a}\cdot(\vec{\nabla}\times\vec{\nabla}f) = 0$ $\Big\}$ Curl of the gradient is identically zero.

for vector $\vec{a}$ = const , we have

$$\vec{\nabla}\cdot\vec{B} = +\frac{\mu_0}{4\pi}\int_V \vec{j}(\vec{r}\,')\cdot\left(\vec{\nabla}\times\vec{\nabla}\frac{1}{\vec{r}-\vec{r}\,'}\right)dV' = 0.$$

(c)



For the magnetic field produced by a circuit C carrying current I, we have

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I\, d\vec{r}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

In this case the current flows across the whole sheet from $-a$ to a, a the strip of width dy' will carry current

$$dI = I\frac{dy'}{2a}$$

Hence, we must replace

$$I\, d\vec{r}\,' \longrightarrow I\frac{dy'}{2a}\vec{k}\, dz'$$ (since the current is along the z axis)

and integrate over the whole strip.

$$\vec{r}\,' = y'\vec{j} + z'\vec{k}\ .$$

Because of the symmetry of the system, the magnetic field does not depend upon z, so we set

$$\vec{r} = x\vec{i} + y\vec{j}.$$

Then $$\vec{r} - \vec{r}\,' = x\vec{i} + (y-y')\vec{j} - z'\vec{k}$$

$$\vec{k} \times (\vec{r} - \vec{r}\,') = \vec{k} \times \left(x\vec{i} + (y-y')\vec{j} - z'\vec{k} \right)$$

$$= x\vec{j} + (y'-y)\,\vec{i}$$

$$|\vec{r} - \vec{r}\,'| = \sqrt{x^2 + (y-y')^2 + z'^2}.$$

Therefore:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I}{2a} \int_{-\infty}^{+\infty}\int_{-a}^{a} \frac{dy'dz'\left(x\vec{j} + (y'-y)\vec{i} \right)}{\left[x^2 + (y-y')^2 + z'^2 \right]^{3/2}}$$

Integration over dz' is done as

$$\int_{-\infty}^{+\infty} \frac{dz'}{(\alpha^2 + z'^2)^{3/2}} = \frac{2}{\alpha^2} \quad \text{where } \alpha^2 = x^2 + (y-y')^2.$$

Hence:

$$\vec{B} = \frac{\mu_0 I}{4\pi a} \int_{-a}^{a} \frac{(y'-y)\vec{i} + x\vec{j}}{x^2 + (y-y')^2}\, dy'$$

Working out the two components separately.

$$B_x = \frac{\mu_0 I}{4\pi a} \int_{-a}^{a} \frac{(y'-y)\,dy'}{x^2 + (y'-y)^2} \qquad (y'-y)dy' = \tfrac{1}{2}\, d\left((y'-y)^2\right) = \tfrac{1}{2}\, d\left((y'-y)^2 + x^2\right)$$

$$= \frac{\mu_0 I}{4\pi a} \frac{1}{2} \int_{-a}^{a} \frac{d\left[x^2 + (y'-y)^2\right]}{x^2 + (y'-y)^2}$$

$$= \frac{\mu_0 I}{8\pi a} \left[\ln\left[x^2 + (y'-y)^2 \right] \right]_{-a}^{a} = \frac{\mu_0 I}{8\pi a} \ln \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2}.$$

$$B_y = \frac{\mu_0 I x}{4\pi a} \int_{-a}^{a} \frac{dy'}{x^2 + (y'-y)^2}$$

Recall:

$$\int \frac{dt}{1+t^2} = \arctan t + C$$

$$= \frac{\mu_0 I x}{4\pi a x^2} \int_{-a}^{a} \frac{dy'}{1 + \left(\frac{y'-y}{x}\right)^2}$$

$$= \frac{\mu_0 I}{4\pi a} \int_{-a}^{a} \frac{d\left(\frac{y'-y}{x}\right)}{1 + \left(\frac{y'-y}{x}\right)^2}$$

$$= \frac{\mu_0 I}{4\pi a} \left[\arctan \frac{y'-y}{x}\right]_{-a}^{a}$$

$$= \frac{\mu_0 I}{4\pi a} \left[\arctan \frac{a-y}{x} - \arctan \frac{-a-y}{x}\right]$$

arctan is an odd function

$$= \frac{\mu_0 I}{4\pi a} \left(\arctan \frac{a-y}{x} + \arctan \frac{a+y}{x}\right).$$