AMA3001 Electromagnetic Theory
Problem sheet 6
SOLUTIONS

① (a) Equations of magnetostatics in the presence of magnetic media are

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (1)$$

$$\vec{\nabla}\times\vec{H} = \vec{j}, \qquad (2)$$

where $\vec{j}$ is the current density of free currents, with the relation

$$\vec{B} = \mu_0(\vec{H}+\vec{M}), \qquad (3)$$

where $\vec{M}$ is the magnetisation.

In the absence of free currents ($\vec{j}=0$),

$$\vec{\nabla}\times\vec{H} = 0,$$

which means that $\vec{H}$ can be represented as

$$\vec{H} = -\vec{\nabla}\varphi_m, \qquad (4)$$

where φ_m is the magnetic scalar potential.

For constant magnetisation, $\vec{M} = \text{const}$, from equations (1) and (3),

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \frac{1}{\mu_0}\vec{\nabla}\cdot\vec{B} - \vec{\nabla}\cdot\vec{M} = 0.$$

Substituting $\vec{H}$ from (4), we have

$$\vec{\nabla}\cdot(\vec{\nabla}\varphi_m) = 0,$$

or $$\underline{\underline{\nabla^2\varphi_m = 0}},$$

so the magnetic potential satisfies Laplace's equation.

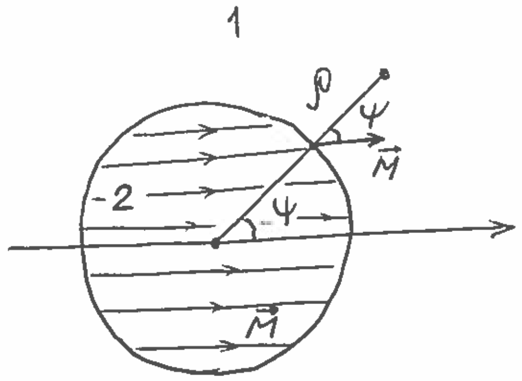
[This is also true for linear isotropic media with constant permeability: $\vec{B}=\mu\vec{H}$ gives $\vec{\nabla}\cdot\vec{H}=0$ by (1).]

(b) In this problem the magnetic scalar potential satisfies Laplace's equation,

$$\nabla^2 \varphi = 0$$

(dropping subscript m, as there is no confusion with the electrostatic potential in this problem sheet). By the symmetry of the system, the magnetic scalar potential does not depend on z, with the z axis chosen as the axis of the cylinder. The general solution of Laplace's equation can then be written as

$$\varphi(\rho, \psi) = C \ln\rho + \sum_{n=-\infty}^{\infty} (A_n \cos n\psi + B_n \sin n\psi)\rho^n .$$



Cross section of the cylinder

We choose the x axis along the magnetisation $\vec{M}$.

Denoting the solution <u>outside</u> ($\rho > a$) by subscript 1, we have:

$$\varphi_1 = C_1 \ln\rho + \sum_{n=-\infty}^{+\infty} (A_n^1 \cos n\psi + B_n^1 \sin n\psi)\rho^n ,$$

and the solution <u>inside</u> ($\rho < a$),

$$\varphi_2 = C_2 \ln\rho + \sum_{n=-\infty}^{+\infty} (A_n^2 \cos n\psi + B_n^2 \sin n\psi)\rho^n .$$

Boundary conditions: $\varphi_1(a, \psi) = \varphi_2(a, \psi)$ (1)

and $-\frac{\partial \varphi_1}{\partial n} + M_{1n} = -\frac{\partial \varphi_2}{\partial n} + M_{2n}$, which is,

in our case $-\left.\frac{\partial \varphi_1}{\partial \rho}\right|_{\rho=a} = -\left.\frac{\partial \varphi_{11}}{\partial \rho}\right|_{\rho=a} + M\cos\psi$ (see diagram). (2)

Since there is no magnetic field at infinity ($\rho \to \infty$), φ_1 should not contain any terms that increase with ρ, so that

$$\varphi_1 = \sum_{n=0}^{\infty} (A_n^1 \cos n\varphi + B_n^1 \sin n\varphi)\rho^{-n}$$

(where we changed $n \to -n$ for convenience, to avoid having coefficients with negative indices).

The magnetic field <u>inside</u> should have no singularity at $\rho = 0$, so

$$\varphi_2 = \sum_{n=0}^{\infty} (A_n^2 \cos n\varphi + B_n^2 \sin n\varphi)\rho^n$$

(with no $\ln\rho$ term or negative powers of ρ).

Using boundary condition (1), we have:

$$\sum_{n=0}^{\infty} (A_n^1 \cos n\varphi + B_n^1 \sin n\varphi)a^{-n} = \sum_{n=0}^{\infty} (A_n^2 \cos n\varphi + B_n^2 \sin n\varphi)a^n.$$

Making the coefficients of $\cos n\varphi$ on both sides equal, and doing the same for $\sin n\varphi$ terms, we have:

$$A_n^1 a^{-n} = A_n^2 a^n \quad (3) \qquad (n \geq 0).$$

$$B_n^1 a^{-n} = B_n^2 a^n \quad (4)$$

Applying boundary condition (2),

$$-\sum_{n=0}^{\infty} (A_n^1 \cos n\varphi + B_n^1 \sin n\varphi)(-n)a^{-n-1} = -\sum_{n=0}^{\infty} (A_n^2 \cos n\varphi + B_n^2 \sin n\varphi)n a^{n-1} + M\cos\varphi.$$

This gives <u>no</u> condition for $n = 0$;

$n = 1$, $\cos\varphi$ terms: $\quad A_1^1 \frac{1}{a^2} = -A_1^2 + M \quad (5')$

$n > 1$, $\cos n\varphi$ terms: $\quad n A_n^1 \frac{1}{a^{n+1}} = -n A_n^2 a^{n-1} \quad (5'')$

sin $n\varphi$ terms: $\quad n B_n^1 a^{-n-1} = -n B_n^2 a^{n-1} \quad (6)$.

From (4), $\quad B_n^1 = B_n^2 a^{2n}$,

and from (6) $\quad B_n^1 = -B_n^2 a^{2n}$ $\quad \Rightarrow \quad B_n^1 = -B_n^1$,

which means that $\quad \underline{B_n^1 = 0}$ and $\underline{B_n^2 = 0}$.

From (3) for $n=0$ $\quad \underline{A_0^1 = A_0^2} \equiv A_0$.

From (3) for $n>0$ $\quad A_n^1 = A_n^2 a^{2n}$

and from (5'') $\quad A_n^1 = -A_n^2 a^{2n} \quad (n>1)$.

Combining these gives $\quad A_n^1 = -A_n^1 \Rightarrow \underline{A_n^1 = 0}$

and $\underline{A_n^2 = 0}$ for $n>1$.

Finally, for $n=1$ from (3):

$$A_1^1 \frac{1}{a^2} = A_1^2 \qquad (7')$$

and from (5')

$$A_1^1 \frac{1}{a^2} = -A_1^2 + M \qquad (7'')$$

Adding these up, we find

$$2A_1^1 \frac{1}{a^2} = M \quad \Rightarrow \quad A_1^1 = \frac{Ma^2}{2}.$$

Subtracting (7'') from (7'):

$$0 = 2A_1^2 - M \quad \Rightarrow \quad A_1^2 = \frac{1}{2} M.$$

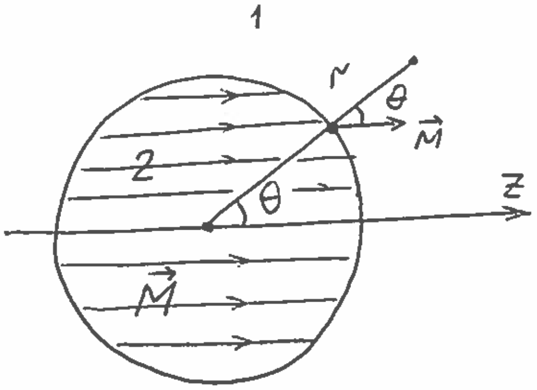
Hence the magnetic scalar potential is:

$$\varphi_2 = A_0 + \frac{1}{2} M \rho \cos\varphi, \qquad \rho < a$$

$$\varphi_1 = A_0 + \frac{Ma^2}{2\rho} \cos\varphi, \qquad \rho > a$$

where we can set A_0 to zero. (Adding a constant to the potential φ does not affect the magnetic field.)

② (a)



This system is axially symmetric, i.e., the solution of Laplace's equation, $\nabla^2 \varphi = 0$, should not depend on ψ if we use spherical coordinates (r, θ, ψ) and direct the z axis along the magnetisation vector.

In this case the general solution of Laplace's equation can be written as an expansion in Legendre polynomials,

$$\varphi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta).$$

Since there is no magnetic field at infinity, the solution valid for $r > a$ (region 1) is

$$\varphi_1 = A_0' + \sum_{n=0}^{\infty} \frac{B_n^1}{r^{n+1}} P_n(\cos\theta). \quad \Big\} \; A_n^1 = 0 \text{ for } n \geq 1.$$

The solution inside the sphere, $r<a$ (region 2) should have no singularity at $r=0$, hence

$$\varphi_2 = \sum_{n=0}^{\infty} A_n^2 r^n P_n(\cos\theta) \quad \Big\} \; B_n^2 = 0 \text{ for all } n \geq 0.$$

At the boundary, $r=a$, these potential should satisfy the boundary conditions:

$$\varphi_1(a,\theta) = \varphi_2(a,\theta), \qquad (1)$$

and

$$-\left.\frac{\partial \varphi_1}{\partial r}\right|_{r=a} = -\left.\frac{\partial \varphi_2}{\partial r}\right|_{r=a} + M\cos\theta. \qquad (2)$$

[The latter follows from the general expression

$$-\frac{\partial \varphi_1}{\partial n} + M_{1n} = -\frac{\partial \varphi_2}{\partial n} + M_{2n} \quad \text{with } \vec{M}_1 = 0,$$

$M_{2n} = M\cos\theta$ - see diagram.]

Using (1) and making the coefficients multiplying $P_n(\cos\theta)$ on both sides equal, we obtain:

$$n=0 \qquad A_0^1 + \frac{B_0^1}{a} = A_0^2 \qquad (3')$$

$$n \geq 1 \qquad \frac{B_n^1}{a^{n+1}} = A_n^2 a^n \qquad (3'')$$

Applying (2)

$$+\sum_{n=0}^{\infty} (n+1) \frac{B_n^1}{a^{n+2}} P_n(\cos\theta) = -\sum_{n=0}^{\infty} A_n^2 n a^{n-1} P_n(\cos\theta) + M\cos\theta .$$

For $n=0$ $\qquad + \frac{B_0^1}{a^2} = 0 \Rightarrow \underline{B_0^1 = 0}$,

hence, from (3'), $\qquad \underline{A_0^1 = A_0^2 \equiv A_0}$.

For $n=1$ $\quad (P_1(\cos\theta) = \cos\theta)$

$$2\frac{B_1^1}{a^3} = -A_1^2 + M . \qquad (4')$$

For $n>1$:

$$(n+1)\frac{B_n^1}{a^{n+2}} = -nA_n^2 a^{n-1} \qquad (4'').$$

This gives $\qquad B_n^1 = -\frac{n}{n+1} A_n^2 a^{2n+1}$,

while from (3') $\qquad B_n^1 = A_n^2 a^{2n+1}$.

$$\Rightarrow \qquad A_n^2 = -\frac{n}{n+1} A_n^2 \Rightarrow \underline{A_n^2 = 0} \text{ and } \underline{B_n^1 = 0} \text{ for } n>1 .$$

From (4') $\qquad A_1^2 + 2\frac{B_1^1}{a^3} = M \qquad (5')$

and from (3''): $\qquad A_1^2 - \frac{B_1^1}{a^3} = 0 . \qquad (5'')$

Subtracting (5'') from (5') gives

$$3\,\frac{B_1'}{a^3} = M \quad \Rightarrow \quad B_1' = \frac{Ma^3}{3}$$

and $A_1^2 = \frac{B_1'}{a^3} = \frac{M}{3}$.

Therefore, the potentials are:

$$\varphi_2 = A_0 + \frac{M}{3} r\cos\theta \qquad \text{(inside the sphere)}$$

$$\varphi_1 = A_0 + \frac{Ma^3}{3}\frac{\cos\theta}{r^2} \qquad \text{(outside the sphere)}.$$

Note that the common constant A_0 can be set to 0, so that

$$\varphi = \frac{M}{3} r\cos\theta \qquad (r<a)$$

$$\varphi = \frac{Ma^3}{3}\frac{\cos\theta}{r^2} \qquad (r>a).$$

(b) The magnetic induction is $\vec{B} = \mu_0(\vec{H}+\vec{M})$
$= -\mu_0\vec{\nabla}\varphi + \mu_0\vec{M}$.

Inside the sphere $\varphi = \frac{M}{3} r\cos\theta = \frac{M}{3} z$

$$\Rightarrow \vec{\nabla}\varphi = \frac{M}{3}\vec{k} = \frac{\vec{M}}{3}$$

Hence, $\vec{B} = -\mu_0\frac{\vec{M}}{3} + \mu_0\vec{M} = \frac{2}{3}\mu_0\vec{M}$. $(r<a)$

Outside the sphere $\vec{M}=0$. Using the expression for the gradient in spherical coordinates,

$$\vec{\nabla}\varphi = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\right)\frac{Ma^3}{3}\frac{\cos\theta}{r^2}$$

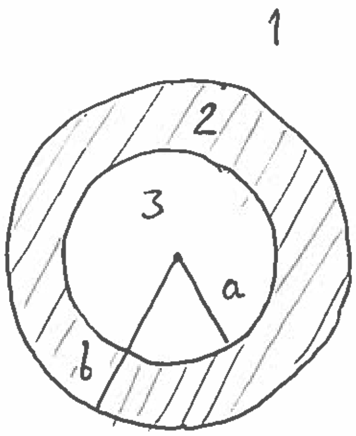
$$= \frac{Ma^3}{3}\left(-\frac{2\hat{r}}{r^3}\cos\theta - \frac{\hat{\theta}}{r^3}\sin\theta\right).$$

The φ component is zero here

Hence, the magnetic induction is

$$\vec{B} = -\mu_0 \vec{\nabla}\varphi = \frac{\mu_0 M a^3}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right). \quad (r>a)$$

③



In this problem we have three regions, and we assume that the magnetic scalar potential in each of them takes the form:

$$1: \quad \varphi_1 = Ar\cos\theta + \frac{B}{r^2}\cos\theta, \quad r>b$$

$$2: \quad \varphi_2 = Cr\cos\theta + \frac{D}{r^2}\cos\theta, \quad a<r<b$$

$$3: \quad \varphi_3 = Er\cos\theta + \frac{F}{r^2}\cos\theta, \quad r<a.$$

Near the origin $(r\to 0)$ the potential must be that of the magnetic dipole $\vec{m}$,

$$\varphi \simeq \frac{\vec{m}\cdot\vec{r}}{4\pi r^3} = \frac{m\cos\theta}{4\pi r^2} \Rightarrow F = \frac{m}{4\pi} \text{ in } \varphi_3.$$

The magnetic field at large distances, $r\to\infty$, must vanish, hence φ_1 should not have terms increasing with r, so $A=0$. Therefore, we have:

$$\varphi_1 = \frac{B}{r^2}\cos\theta$$

$$\varphi_2 = Cr\cos\theta + \frac{D}{r^2}\cos\theta$$

$$\varphi_3 = Er\cos\theta + \frac{m}{4\pi r^2}\cos\theta$$

There are 4 boundary conditions to satisfy:

$$\varphi_1(b) = \varphi_2(b) \quad (1), \qquad \varphi_3(a) = \varphi_2(a) \quad (2)$$

$$\mu_0 \left.\frac{\partial\varphi_1}{\partial r}\right|_{r=b} = \mu \left.\frac{\partial\varphi_2}{\partial r}\right|_{r=b} \quad (3), \qquad \mu \left.\frac{\partial\varphi_2}{\partial r}\right|_{r=a} = \mu_0 \left.\frac{\partial\varphi_3}{\partial r}\right|_{r=a} \quad (4).$$

From (1): $$\frac{B}{b^2} = Cb + \frac{D}{b^2} \qquad (1')$$

From (2): $$Ca + \frac{D}{a^2} = Ea + \frac{m}{4\pi a^2} \qquad (2')$$

From (3): $$-\mu_0 \frac{2B}{b^3} = \mu\left(C - \frac{2D}{b^3}\right) \qquad (3')$$

From (4): $$\mu\left(C - \frac{2D}{a^3}\right) = \mu_0\left(E - \frac{2m}{4\pi a^3}\right) \qquad (4')$$

$$(1') \iff C + \frac{D}{b^3} = \frac{B}{b^3} \qquad (1'')$$

$$(3') \iff C - \frac{2D}{b^3} = -\frac{\mu_0}{\mu}\frac{2B}{b^3} \qquad (3'')$$

Subtracting the latter from the former, we find

$$\frac{3D}{b^3} = \left(1 + \frac{2\mu_0}{\mu}\right)\frac{B}{b^3}$$

$$\Rightarrow \quad D = \frac{1}{3}\left(1 + \frac{2\mu_0}{\mu}\right)B . \qquad (5')$$

Multiplying (1'') by 2 and adding to (3''), we have

$$3C = \left(1 - \frac{\mu_0}{\mu}\right)\frac{2B}{b^3}$$

$$\Rightarrow \quad C = \left(1 - \frac{\mu_0}{\mu}\right)\frac{2B}{3b^3} . \qquad (5'')$$

From (2'):

$$E + \frac{m}{4\pi a^3} = C + \frac{D}{a^3}$$

From (4'):

$$E - \frac{2m}{4\pi a^3} = \frac{\mu}{\mu_0}\left(C - \frac{2D}{a^3}\right)$$

Subtracting the latter from the former, we have:

$$\frac{3m}{4\pi a^3} = \left(1 - \frac{\mu}{\mu_0}\right)C + \left(1 + \frac{2\mu}{\mu_0}\right)\frac{D}{a^3} .$$

Substituting C and D from (5') and (5'') gives:

$$\left(1-\frac{\mu}{\mu_0}\right)\left(1-\frac{\mu_0}{\mu}\right)\frac{2B}{3b^3}+\left(1+\frac{2\mu}{\mu_0}\right)\frac{1}{a^3}\frac{1}{3}\left(1+\frac{2\mu_0}{\mu}\right)B=\frac{3m}{4\pi a^3}$$

$$\frac{B}{3}\left[\left(1-\frac{\mu}{\mu_0}\right)\left(1-\frac{\mu_0}{\mu}\right)\frac{2}{b^3}+\left(1+\frac{2\mu}{\mu_0}\right)\left(1+\frac{2\mu_0}{\mu}\right)\frac{1}{a^3}\right]=\frac{3m}{4\pi a^3}$$

Multiplying both sides by $\mu_0\mu a^3 b^3$ gives

$$\frac{B}{3}\left[2(\mu_0-\mu)(\mu-\mu_0)a^3+(\mu_0+2\mu)(\mu+2\mu_0)b^3\right]=\frac{3m\mu_0\mu b^3}{4\pi}$$

$$\Rightarrow \quad B=\frac{9m\mu_0\mu b^3}{4\pi\left[(\mu_0+2\mu)(\mu+2\mu_0)b^3-2(\mu-\mu_0)^2a^3\right]}.$$

The potential outside the system ($r>b$) is

$$\varphi_1=\frac{B}{r^2}\cos\theta ,$$

and coincides with that of magnetic dipole m':

$$\varphi_1=\frac{m'\cos\theta}{4\pi r^2}$$

for $\quad \frac{m'}{4\pi}=B$.

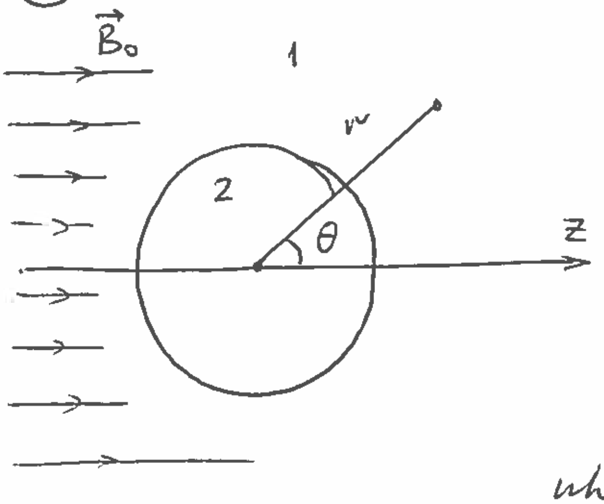
Hence, the magnetic moment of the whole system

$$m'=4\pi B=\frac{9m\mu_0\mu b^3}{(\mu_0+2\mu)(\mu+2\mu_0)b^3-2(\mu-\mu_0)^2a^3}.$$

It is in the same direction as $\vec{m}$, so we can write:

$$\vec{m}'=\frac{9m\mu_0\mu b^3\,\vec{m}}{(\mu_0+2\mu)(\mu+2\mu_0)b^3-2(\mu-\mu_0)^2a^3}.$$

④ (a)



Using the magnetic scalar potential φ, we have

$$\vec{H} = -\vec{\nabla}\varphi$$

and $\vec{B} = -\mu_0 \vec{\nabla}\varphi$ (outside the sphere).

At large distances $\vec{B} \simeq \vec{B}_0$, where $\vec{B}_0 = B_0 \vec{k}$ (along the z axis).

Hence, we must have

$$\varphi \simeq -\frac{B_0}{\mu_0} z = -\frac{B_0}{\mu_0} r\cos\theta \quad \text{at } r \to \infty.$$

The magnetic scalar potential satisfies Laplace's equation and its axially symmetric (independent of ψ) solutions are:

$$\varphi_1(r,\theta) = \sum_{n=0}^{\infty} \left(A_n^1 r^n + \frac{B_n^1}{r^{n+1}}\right) P_n(\cos\theta), \quad r>a$$

$$\varphi_1(r,\theta) = \sum_{n=0}^{\infty} \left(A_n^2 r^n + \frac{B_n^2}{r^{n+1}}\right) P_n(\cos\theta), \quad r<a$$

The boundary conditions are:

at $r \to \infty$ $\qquad \varphi_1 \simeq -\frac{B_0}{\mu_0} r\cos\theta$, $\qquad$ (1)

for $r \to 0$ $\qquad \varphi_2$ is finite. $\qquad$ (2)

At $r=a$: $\qquad \varphi_1(a,\theta) = \varphi_2(a,\theta)$ $\qquad$ (3)

$$\mu_0 \left.\frac{\partial \varphi_1}{\partial r}\right|_{r=a} = \mu \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=a} \qquad (4)$$

From (1), $A_n^1 = 0$ for $n>1$, $A_1^1 = -\frac{B_0}{\mu_0}$.

From (2), $B_n^2 = 0$ for all n.

Thus, we have:

$$\varphi_1 = A_0^1 - \frac{B_0}{\mu_0} r\cos\theta + \sum_{n=0}^{\infty} \frac{B_n^1}{r^{n+1}} P_n(\cos\theta)$$

$$\varphi_2 = \sum_{n=0}^{\infty} A_n^2 r^n P_n(\cos\theta) \quad .$$

From (3), making the coefficients of $P_n(\cos\theta)$ on both sides equal, we obtain:

$$n=0: \quad A_0^1 + \frac{B_0^1}{a} = A_0^2 \quad [\Rightarrow A_0^1 = A_0^2 \equiv A_0] \quad (5')$$

$$n=1: \quad -\frac{B_0}{\mu_0} a + \frac{B_1^1}{a^2} = A_1^2 a \quad (5'')$$

$$n>1: \quad \frac{B_n^1}{a^{n+1}} = A_n^2 a^n \quad . \quad (5''')$$

(using this gives)

Similarly from (4): $n=0$: $\mu_0\left(-\frac{B_0^1}{a^2}\right) = 0 \Rightarrow \underline{B_0^1 = 0}$

$$n=1 \quad \mu_0\left(-\frac{B_0}{\mu_0} - \frac{2B_1^1}{a^3}\right) = \mu A_1^2 \quad (6')$$

$$n>1 \quad -\mu_0 (n+1)\frac{B_n^1}{a^{n+2}} = \mu n A_n^2 a^{n-1} \quad (6'')$$

From (5'''), $\quad B_n^1 = A_n^2 a^{2n+1}$,

and from (6''), $\quad B_n^1 = -\frac{\mu n}{\mu_0 (n+1)} A_n^2 a^{2n+1}$,

$$\Rightarrow \quad A_n^2 = -\frac{\mu n}{\mu_0(n+1)} A_n^2 \Rightarrow \underline{A_n^2 = 0},$$

and $\underline{B_n^1 = 0}$ for $n > 1$.

From (5'') $\quad A_1^2 - \frac{B_1^1}{a^3} = -\frac{B_0}{\mu_0}, \quad (7')$

and from (6') $\quad A_1^2 + \frac{\mu_0}{\mu}\frac{2B_1^1}{a^3} = -\frac{B_0}{\mu} . \quad (7'')$

Subtracting the former from the latter, we find

$$\left(1 + \frac{2\mu_0}{\mu}\right) \frac{B_1^1}{a^3} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) B_0$$

$$\Rightarrow \quad B_1^1 = \frac{(\mu - \mu_0)}{\mu_0(\mu + 2\mu_0)} B_0 a^3$$

Multiplying (7') by $\frac{2\mu_0}{\mu}$ and adding it to (7'') give

$$A_1^2 \left(1 + \frac{2\mu_0}{\mu}\right) = -3 \frac{B_0}{\mu}$$

$$\Rightarrow \quad A_1^2 = -3 \frac{B_0}{\mu + 2\mu_0}$$

Setting the arbitrary constant $A_0 = 0$, we then have:

$$\varphi_2 = -\frac{3B_0}{\mu + 2\mu_0} r\cos\theta \quad \text{(inside the sphere)},$$

$$\varphi_1 = -\frac{B_0}{\mu_0} r\cos\theta + \frac{\mu - \mu_0}{\mu_0(\mu + 2\mu_0)} \frac{B_0 a^3}{r^2} \cos\theta$$

(outside the sphere).

(b) The magnetic induction $\vec{B}$ is found

from $\vec{B} = -\mu_0 \vec{\nabla}\varphi$ (outside)

and $\vec{B} = -\mu \vec{\nabla}\varphi$ (inside)

Noting that $r\cos\theta = z$, $\vec{\nabla}(r\cos\theta) = \vec{\nabla} z = \vec{k}$.

Hence, inside the sphere,

$$\vec{B} = \mu \frac{3B_0}{\mu + 2\mu_0} \vec{k} = \frac{3\mu}{\mu + 2\mu_0} \vec{B}_0 .$$

In spherical coordinates

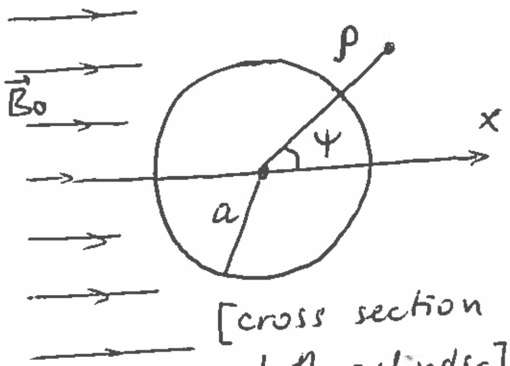
$$\vec{\nabla}\left(\frac{\cos\theta}{r^2}\right) = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\cos\theta}{r^2}\right) = -\frac{2\hat{r}}{r^3}\cos\theta - \frac{\hat{\theta}\sin\theta}{r^3}$$

Hence, the magnetic induction <u>outside the sphere</u>

is $$\vec{B} = B_0 \vec{k} + \frac{\mu - \mu_0}{\mu + 2\mu_0} B_0 a^3 \left(-\frac{2\hat{r}}{r^3} \cos\theta - \frac{\hat{\theta}}{r^3} \sin\theta \right),$$

or $$\vec{B} = \vec{B}_0 + \frac{\mu - \mu_0}{\mu + 2\mu_0} B_0 \frac{a^3}{r^3} \left(2 \cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right).$$

(5) (a) The magnetic scalar potential satisfied Laplace's equation both inside and outside the cylinder. If we use cylindrical coordinates (ρ, ψ, z) and direct the z axis along the axis of the cylinder, the potential does not depend on z and the solution can be written as



[cross section of the cylinder]

$$\varphi(\rho, \psi) = C \ln\rho + \sum_{n=-\infty}^{+\infty} (A_n \cos n\psi + B_n \sin n\psi) \rho^n.$$

At large distances ρ from the cylinder the field is uniform in the x direction (see diagram),

$$\vec{B} \simeq \vec{B}_0 = B_0 \vec{i}$$

and the potential which satisfies $\vec{B} = -\mu_0 \vec{\nabla} \varphi$ must then behave as

$$\varphi \simeq -\frac{B_0}{\mu_0} x = -\frac{B_0}{\mu_0} \rho \cos\psi.$$

The solution <u>outside</u> the cylinder should have no terms growing faster than this with ρ. Hence, here

$$\varphi_1 = C_1 \ln\rho - \frac{B_0}{\mu_0} \rho \cos\psi + \sum_{n=0}^{\infty} (A_n' \cos n\psi + B_n' \sin n\psi) \times \rho^{-n}.$$

Inside the cylinder, there is no singularity at $\rho \to 0$ so the potential must have the form

$$\varphi_2 = \sum_{n=0}^{\infty} (A_n^2 \cos n\psi + B_n^2 \sin n\psi)\rho^n .$$

Boundary conditions:

$$\varphi_1(a,\psi) = \varphi_2(a,\psi) \qquad (1)$$

$$\mu_0 \left.\frac{\partial \varphi_1}{\partial \rho}\right|_{\rho=a} = \mu \left.\frac{\partial \varphi_2}{\partial \rho}\right|_{\rho=a} \qquad (2).$$

From (1), comparing the coefficients which multiply $\cos n\psi$ and $\sin n\psi$ (and the ψ-independent term, $n=0$), we obtain:

$n=0$: $\quad C_1 \ln a + A_0^1 = A_0^2 \qquad (3')$

$n=1$: $\quad -\frac{B_0}{\mu_0} a + A_1^1 \frac{1}{a} = A_1^2 a \qquad (3'')$ } $\cos\psi$ terms

$n>1$: $\quad A_n^1 \frac{1}{a^n} = A_n^2 a^n \qquad (3''')$

$\sin n\psi$ terms (all $n>0$): $\quad B_n^1 \frac{1}{a^n} = B_n^2 a^n . \qquad (3^{iv})$

From boundary condition (2):

$n=0$ $\quad \mu_0 C_1 \frac{1}{a} = 0 \qquad (4')$

$n=1$ $\quad \mu_0\left(-\frac{B_0}{\mu_0} - A_1^1 \frac{1}{a^2}\right) = \mu A_1^2 \qquad (4'')$

$n>1$ $\quad \mu_0(-n) A_n^1 \frac{1}{a^{n+1}} = \mu n A_n^2 a^{n-1} \qquad (4''')$

} $\cos\psi$ and $\cos n\psi$ terms

$\sin n\psi$ terms: $\quad \mu_0(-n) B_n^1 \frac{1}{a^{n+1}} = \mu n B_n^2 a^{n-1} \qquad (4^{iv})$

From (4') $C_1 = 0$. Then (3') gives

$$A_0^1 = A_0^2 \equiv A_0 .$$

From (3''') and (4''')

$$A_n^1 = A_n^2 a^{2n} \quad \text{and} \quad A_n^1 = -\frac{\mu}{\mu_0} A_n^2 a^{2n},$$

so that $$A_n^2 = -\frac{\mu}{\mu_0} A_n^2 \Rightarrow \underline{A_n^2 = 0}$$

and $\underline{A_n^1 = 0}$ for $n > 1$.

From (3^{iv}) and (4^{iv}),

$$B_n^1 = B_n^2 a^{2n} \quad \text{and} \quad B_n^1 = -\frac{\mu}{\mu_0} B_n^2 a^{2n}$$

$$\Rightarrow B_n^2 = -\frac{\mu}{\mu_0} B_n^2 \Rightarrow \underline{B_n^2 = 0} \text{ and } \underline{B_n^1 = 0}$$

for all $n > 0$.

From (3'') and (4''):

$$A_1^2 - \frac{A_1^1}{a^2} = -\frac{B_0}{\mu_0}, \qquad (5')$$

$$A_1^2 + \frac{\mu_0}{\mu}\frac{A_1^1}{a^2} = -\frac{B_0}{\mu} \qquad (5'')$$

Subtracting (5') from (5''),

$$\left(1 + \frac{\mu_0}{\mu}\right)\frac{A_1^1}{a^2} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) B_0$$

$$\Rightarrow \underline{A_1^1 = \frac{\mu - \mu_0}{\mu_0(\mu + \mu_0)} B_0 a^2} .$$

Multiplying (5') by μ_0/μ and adding to (5''),

$$\left(1 + \frac{\mu_0}{\mu}\right) A_1^2 = -\frac{2B_0}{\mu} \Rightarrow \underline{A_1^2 = -\frac{2B_0}{\mu + \mu_0}} .$$

Setting the arbitrary constant A_0 to zero we obtain:

$$\varphi_2 = -\frac{2B_0}{\mu+\mu_0}\rho\cos\varphi \qquad \text{(inside the cylinder)}$$

$$\varphi_1 = -\frac{B_0}{\mu_0}\rho\cos\varphi + \frac{\mu-\mu_0}{\mu_0(\mu+\mu_0)}B_0 a^2\frac{\cos\varphi}{\rho}$$

(outside the cylinder).

(b) The magnetic induction $\vec{B}$ inside the cylinder is given by $\vec{B} = -\mu\vec{\nabla}\varphi$.

$$\vec{\nabla}(\rho\cos\varphi) = \vec{\nabla}x = \vec{i}.$$

Therefore,

$$\vec{B} = \frac{2\mu B_0}{\mu+\mu_0}\vec{i} = \frac{2\mu}{\mu+\mu_0}\vec{B}_0, \qquad \text{(inside the cylinder).}$$

Similarly, outside the cylinder,

$$\vec{B} = -\mu_0\vec{\nabla}\varphi.$$

In cylindrical coordinates,

$$\vec{\nabla}\frac{\cos\varphi}{\rho} = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\varphi}}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\frac{\cos\varphi}{\rho}\right)$$

$$= -\frac{\hat{\rho}}{\rho^2}\cos\varphi - \frac{\hat{\varphi}}{\rho^2}\sin\varphi.$$

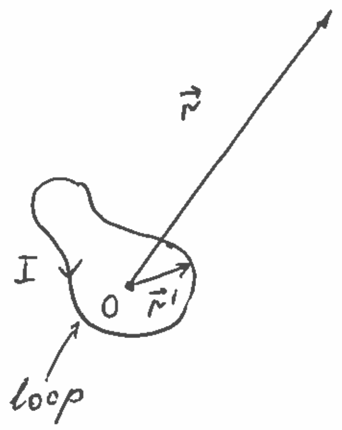
Therefore,

$$\vec{B} = B_0\vec{i} - \mu_0\left(\frac{\mu-\mu_0}{\mu_0(\mu+\mu_0)}B_0a^2\left(-\frac{\hat{\rho}}{\rho^2}\cos\varphi - \frac{\hat{\varphi}}{\rho^2}\sin\varphi\right)\right)$$

$$\Rightarrow \quad \vec{B} = \vec{B}_0 + \frac{\mu-\mu_0}{\mu+\mu_0}B_0\frac{a^2}{\rho^2}\left(\cos\varphi\,\hat{\rho} + \sin\varphi\,\hat{\varphi}\right).$$

(outside the cylinder).

⑥ (a)



At distances $r \gg r'$, we can expand $\frac{1}{|\vec{r}-\vec{r}'|}$ to first order:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \vec{\nabla}\frac{1}{r}\cdot(-\vec{r}') + \dots$$

$$\simeq \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3}$$

The vector potential $A(\vec{r}) = \frac{\mu_0}{4\pi} I \oint_C \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}$ then becomes:

$$A(\vec{r}) = \frac{\mu_0}{4\pi} I \oint \left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3}\right] d\vec{r}'$$

$$= \frac{\mu_0}{4\pi}\frac{I}{r}\oint_C d\vec{r} + \frac{\mu_0}{4\pi}\frac{I}{r^3}\oint_C (\vec{r}\cdot\vec{r}')\, d\vec{r}' \qquad (1)$$

In the first term $\oint_C d\vec{r}' = 0$ for any closed loop C.
To transform the 2nd integral, we use two identities:

$$d\left[(\vec{r}\cdot\vec{r}')\vec{r}'\right] = (\vec{r}\cdot d\vec{r}')\,\vec{r}' + (\vec{r}\cdot\vec{r}')\, d\vec{r}'$$

$$\vec{r}\times(\vec{r}'\times d\vec{r}') = \vec{r}'(\vec{r}\cdot d\vec{r}') - d\vec{r}'(\vec{r}\cdot\vec{r}'),$$

which give:

$$d\left[(\vec{r}\cdot\vec{r}')\vec{r}'\right] - \vec{r}\times(\vec{r}'\times d\vec{r}') = 2(\vec{r}\cdot\vec{r}')\, d\vec{r}',$$

or
$$(\vec{r}\cdot\vec{r}')\, d\vec{r}' = \frac{1}{2}\left\{ d\left[(\vec{r}\cdot\vec{r}')\vec{r}'\right] + (\vec{r}'\times d\vec{r}')\times\vec{r}\right\}$$

The 2nd term in (1) then gives:

$$A(\vec{r}) = \underbrace{\frac{\mu_0}{8\pi}\frac{I}{r^3}\oint_C d\left[(\vec{r}\cdot\vec{r}')\vec{r}'\right]}_{0 \text{ for any closed loop } C} + \frac{\mu_0}{4\pi}\frac{I}{r^3}\frac{1}{2}\oint_C \vec{r}'\times d\vec{r}' \times\vec{r}$$

0 for any closed loop C.

Introducing the <u>vector area</u> of the loop,

$$\vec{a} = \frac{1}{2} \oint_C \vec{r}\,' \times d\vec{r}\,'$$

and the <u>magnetic moment</u> $\vec{m} = I\vec{a}$,

we have:

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} .$$

(b) A volume dV' of the magnetic material has the magnetic dipole moment $\vec{M}\,dV'$. If this volume element has position $\vec{r}\,'$, the vector potential due to it, at point $\vec{r}$ is

$$\frac{\mu_0}{4\pi} \frac{\vec{M}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} dV' ,$$

and the vector potential of the whole piece of material is given by the volume integral over the whole volume of the material:

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{M}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} dV' . \qquad (1)$$

Using the identity

$$\vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} ,$$

we can write (1) as

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{r}\,') \times \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|} dV'$$

We now use the vector identity

$$\vec{\nabla} \times (\vec{a} f) = (\vec{\nabla} \times \vec{a}) f - \vec{a} \times (\vec{\nabla} f) ,$$

and obtain:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\nabla}'\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV' - \frac{\mu_0}{4\pi}\int_V \vec{\nabla}'\times\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV'$$

The 2nd term can be transformed into the surface integral using the identity

$$\int_V \vec{\nabla}\times\vec{a}\,dV = -\oint_S \vec{a}\times d\vec{S} \qquad (*)$$

[Proof: multiplying the left-hand side with an arbitrary constant vector $\vec{t}$, we have:

$$\vec{t}\cdot\int_V \vec{\nabla}\times\vec{a}\,dV = \int_V \vec{t}\cdot(\vec{\nabla}\times\vec{a})\,dV = \int_V \vec{\nabla}\cdot(\vec{a}\times\vec{t})\,dV$$

$$= \oint_S (\vec{a}\times\vec{t})\cdot d\vec{S} = \left(-\oint \vec{a}\times d\vec{S}\right)\cdot\vec{t},$$

and since $\vec{t}$ is arbitrary, (*) follows.]

Therefore:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\nabla}'\times M(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{M}(\vec{r}')\times d\vec{S}'}{|\vec{r}-\vec{r}'|}.$$

The first term describes the vector potential produced by the volume currents with density

$$\vec{j}_m = \vec{\nabla}\times\vec{M}.$$

In the second term, $\vec{M}(\vec{r}')\times d\vec{S}' = \vec{M}\times\vec{n}\,dS'$ and it describes the contribution of surface currents with the surface density

$$\vec{J}_m = \vec{M}\times\vec{n}.$$

[The surface current density is the current crossing a unit length

(c) i. The total current density $\vec{j}_{tot}$ contains the contribution of free currents ($\vec{j}$) and magnetisation currents ($\vec{j}_m$).

The differential form of Biot-Savart's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_{tot} ,$$

then gives:

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_m)$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \vec{j} + \vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{j} .$$

Introducing the magnetic field intensity $\vec{H}$,

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} ,$$

we have:

$$\vec{\nabla} \times \vec{H} = \vec{j} .$$

ii. For a wide class of materials, the magnetisation is proportional to $\vec{H}$,

$$\vec{M} = \chi_m \vec{H} ,$$

where χ_m is the magnetic susceptibility. Materials with (small) negative χ_m are called diamagnetics. Materials with (small) positive χ_m are called paramagnetics. Another class of materials called ferromagnetics can have large magnetisations. However, they are typically not linear and their magnetisation (and induction $\vec{B} = \mu_0(\vec{H} + \vec{M}) \equiv \mu \vec{H}$) depend on the history of the sample.