Electromagnetic Theory AMA3001 **Problem sheet 8**

Maxwell's equations, electromagnetic energy and potentials.

Homework

1. (a) A medium has constant permittivity ε and conductivity σ. If $\rho_0(\boldsymbol{r})$ is the density of free charge inside the medium at time $t = 0$, show that at time t the density is

$$\rho(\boldsymbol{r}, t) = \rho_0(\boldsymbol{r}) e^{-t/\tau},$$

where $\tau = \varepsilon/\sigma$. (Note that this result is independent of any other electromagnetic disturbance that may be taking place simultaneously.)

[Hint: use the equation of continuity, Ohm's law and Maxwell's first equation.]

(b) An uncharged sphere of radius R composed of material of permittivity ε and conductivity σ is placed in vacuum. At time $t = 0$ an amount Q_0 of charge is distributed uniformly over a spherical shell at distance a ($a < R$) from the centre of the sphere.

i. Prove that at time t this charge has decreased to the value

$$Q(t) = Q_0 e^{-\sigma t/\varepsilon}.$$

How has the charge distribution elsewhere in the sphere changed?

ii. Find the amount of energy dissipated in the time interval $t = 0$ to $t = T$ and verify that it is exactly equal to the electrostatic energy difference calculated assuming that the charge distributions at $t = 0$ and $t = T$ are static.

2. A straight metal wire of conductivity σ and of circular cross section of radius a, carries a steady current I. Determine the direction and magnitude of the Poynting vector at the surface of the wire. Integrate the normal component of the Poynting vector over the surface of the wire for a segment of length L, and compare the result with the Joule heat produced in the segment.

3. (a) Using Maxwell's equations for a linear, isotropic medium, show how the quantities

$$w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \quad \text{and} \quad \boldsymbol{j} \cdot \boldsymbol{E}$$

can be interpreted in terms of energy balance within a volume.

(b) Two parallel circular plates of radius a, separated by distance $d \ll a$, form a capacitor containing a linear dielectric. The capacitor is charged by a constant current I from zero at time $t = 0$. Neglecting edge effects and assuming the dielectric is a perfect insulator, it can be shown that the fields inside the capacitor are given by

$$\boldsymbol{D} = \frac{It}{\pi a^2} \boldsymbol{k} \quad \text{and} \quad \boldsymbol{H} = \frac{I\rho}{2\pi a^2} \hat{\boldsymbol{\psi}},$$

where (ρ, ψ, z) are cylindrical polar coordinates centred on one of the plates.

i. Verify that these fields satisfy Maxwell's equations.

ii. Show that the energy balance equation is satisfied by the dielectric.

[Exam 2003 Question 4]

4. The vector and scalar potentials in Lorenz gauge ($\nabla\cdot\boldsymbol{A}+\varepsilon\mu\partial\phi/\partial t=0$) satisfy the equations

$$\nabla^2\boldsymbol{A}-\varepsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2}=-\mu\boldsymbol{j}\quad\text{and}\quad\nabla^2\phi-\varepsilon\mu\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\varepsilon}.$$

   (a) Verify that if $f(u)$ is a function of the variable $u=t-r/v$ where $v=(\varepsilon\mu)^{-1/2}$, then

$$\boldsymbol{A}=\frac{f'(u)}{vr}\boldsymbol{k}\quad\text{and}\quad\phi=\left[\frac{f'(u)}{r}+\frac{vf(u)}{r^2}\right]\frac{z}{r},\tag{1}$$

   where $\boldsymbol{k}$ is the unit vector in the z direction, satisfy the differential equations for the vector and scalar potentials, respectively, in a region of space in which the current and charge densities are zero and $r\neq 0$.

   (b) Verify that the potentials (1) also satisfy the Lorenz condition. [To calculate $\nabla\cdot\boldsymbol{A}$ in spherical coordinates, use $\boldsymbol{k}=\cos\theta\,\hat{\boldsymbol{r}}-\sin\theta\,\hat{\boldsymbol{\theta}}$ to find the components A_r and A_θ.]

   (c) Using spherical coordinates, obtain expressions for the magnetic induction $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ corresponding to these potentials.

Further Examples

1. A parallel plate condenser with plates having the shape of circular discs has the region between its plates filled with a dielectric of permittivity ε. The dielectric is imperfect, having a conductivity σ. The capacitance of the condenser is C. The condenser is charged to a potential difference $\Delta\phi$ and isolated.

   (a) Find the charge on the condenser as a function of time.

   (b) Find the displacement current in the dielectric.

   (c) Find the magnetic field in the dielectric.

2. The vector potential for an electromagnetic field in vacuum satisfies the equations

$$\nabla\cdot\boldsymbol{A}=0\quad\text{and}\quad\nabla^2\boldsymbol{A}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}=0,$$

   where $c=(\varepsilon_0\mu_0)^{-1/2}$, and the scalar potential is zero.

   (a) Show that the field vectors derived from these potentials satisfy Maxwell's equations for zero charge and current densities.

   (b) Show that a vector potential of the form $\boldsymbol{A}=\boldsymbol{A}(q)$, where $q=\boldsymbol{n}\cdot\boldsymbol{r}-ct$ and $\boldsymbol{n}$ is a unit vector, satisfies these equations provided $\boldsymbol{n}\cdot(d\boldsymbol{A}/dq)=0$.

   (c) Determine $\boldsymbol{B}$ and $\boldsymbol{E}$ for the potential from (b) and show that

$$\boldsymbol{B}=\frac{1}{c}\boldsymbol{n}\times\boldsymbol{E}\quad\text{and}\quad\boldsymbol{n}\cdot\boldsymbol{E}=0.$$

3. Starting from Maxwell's equations, establish the boundary conditions (in the usual notation)

$$\begin{aligned}E_{1t}-E_{2t}&=0,\\ D_{1n}-D_{2n}&=\sigma,\\ B_{1n}-B_{2n}&=0,\\ (\boldsymbol{H}_1-\boldsymbol{H}_2)\cdot\boldsymbol{t}&=(\boldsymbol{J}\times\boldsymbol{n})\cdot\boldsymbol{t},\end{aligned}$$

   at the interface between different dielectric and magnetic media.