# Partial Differential Equations AMA3006

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*Gleb Gribakin*
*Department of Applied Mathematics*
*and Theoretical Physics*
*Queen's University Belfast*

# Contents

# 1 Introduction. Examples of partial differential equations

## 1.1 Terminology

A *partial differential equation* (PDE) is a relation of the form

$$F(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{xy}, \dots) = 0, \tag{1.1}$$

where $u(x, y, \dots)$ is a function of *independent variables* $x$, $y$, …, such that (1.1) is identically satisfied, and

$$u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial x}, \qquad \dots \tag{1.2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \dots \tag{1.3}$$

are its *derivatives.*

The *order* of a PDE is the order of the highest derivative it contains.

A PDE is *linear* if $F$ is a linear function of $u$, $u_x$, $u_y$, …, $u_{xx}$, $u_{xy}$, …. A linear PDE is *homogeneous* if

$$F(x, y, \dots, Cu, Cu_x, \dots, Cu_{xx}, \dots) = CF(x, y, \dots, u, u_x, \dots, u_{xx}, \dots).$$

Otherwise, it is *inhomogeneous.*

Examples.

1. $\dfrac{du}{dx} + p(x) = q(x)$ for $u(x)$ is a 1st-order linear inhomogeneous ordinary differential equation (ODE); for $q = 0$ it is homogeneous.[1]
2. $au'' + bu' + cu = 0$ for $u(x)$ is a 2nd-order linear homogeneous ODE with constant coefficients; $au'' + bu' + cu = f(x)$ is inhomogeneous.[1]
3. $xy\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = x^2$ for $u(x, y)$ is a 1st-order, nonlinear PDE.
4. $\dfrac{\partial^2 u}{\partial x^2} - xy\dfrac{\partial u}{\partial y} = 0$ for $u(x, y)$ is 2nd-order linear homogeneous.
5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{a^4}\dfrac{\partial^2 u}{\partial t^2} = q(x, t)$ for $u(x, t)$ is 4th-order linear inhomogeneous.

   [It describes the shape of a beam under normal stress $q(x, t)$.]

[1] See Vector Algebra and Dynamics AMA1001 for methods of solving.

6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ for $u(x, y, z)$ is 2nd-order linear homogeneous.

   This is Laplace's equation. It is important in electrostatics, fluid dynamics, etc. It can be written in compact form as $\nabla^2 u = 0$, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.4}$$

   is the Laplacian.[2] The inhomogeneous version of Laplace's equation,

$$\nabla^2 u = f(x, y, z)$$

   is known as Poisson's equation.

## 1.2 Variety of solutions

"General solutions" contain arbitrary functions with one independent variable less than $u$.

<u>Examples of equations and solutions.</u>

1. Equation: $u_y = 0$ for $u(x, y)$.

   Solution: $u = w(x)$, where $w$ is an arbitrary function.

2. Equation: $u_{xy} = 0$ for $u(x, y)$.

   Solution: $u = v(x) + w(y)$, where $v$ and $w$ are arbitrary functions.

3. Equation: $u_{xy} = f(x, y)$ for $u(x, y)$.

   Solution: $u = \int_{x_0}^{x} \int_{y_0}^{y} f(x, y) dy dx + v(x) + w(y)$.

4. Equation: $u_x = u_y$ for $u(x, y)$.

   Using new variables, $\xi = x + y$, $\eta = x - y$,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta},$$

   and the equation is reduced to $u_\eta = 0$.

   Solution: $u = w(\xi)$, or $u = w(x + y)$, where $w$ is an arbitrary function.

5. Equation: $u_{xx} - u_{yy} = 0$ for $u(x, y)$.

   The equation can be written as

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0.$$

[2] Recall the basics of the vector field theory: the nabla operator, $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the $x$, $y$ and $z$ axes; $\nabla \cdot \nabla = \nabla^2$ is the Laplacian; $\nabla u = \operatorname{grad} u$, $\nabla \cdot \mathbf{A} = \operatorname{div} \mathbf{A}$, $\nabla \times \mathbf{A} = \operatorname{curl} \mathbf{A}$ (introduced in AMA1002, used in AMA2005).

From example 4 above, using $\xi = x + y$ and $\eta = x - y$,

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta},$$

and the differential equation becomes

$$4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}u = 0.$$

From example 2, we have $u = w(\xi) + v(\eta)$.

Solution: $u = w(x+y)+v(x-y)$, where $v$ and $w$ are arbitrary functions.

<u>Exercise.</u> Show that the solution of $u_{tt} - c^2 u_{xx} = 0$ for $u(x,t)$ is

$$u = w(x + ct) + v(x - ct),$$

where $v$ and $w$ are arbitrary functions. This is known as *d'Alembert's solution* of the wave equation (see Sec. 1.3).

[Hint: use new variables $\xi = x + ct$ and $\eta = x - ct$.]

## 1.3 Wave equation

Using Newton's 2nd law for a small segment of a string, we show that the displacement $u(x,t)$ obeys the *wave equation*,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.5}$$

where $c = \sqrt{T/\rho}$ is the wave *speed*, $T$ is the tension along the string, and $\rho$ is the linear mass density.

If an external force of $f(x,t)$ (per unit length) acts perpendicular to the string, then

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{f(x,t)}{\rho}. \tag{1.6}$$

The wave equation in two dimensions (membrane)

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \tag{1.7}$$

and in three dimensions (sound in air, electromagnetic waves),

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0. \tag{1.8}$$

## 1.4 Heat equation

Fourier's law of heat conduction determines the heat flux density,

$$\mathbf{j}_q = -\kappa \nabla T, \tag{1.9}$$

where $j_q$ is the heat energy that flows through a unit area perpendicular to $\mathbf{j}_q$ in unit time, $T$ is the temperature, and $\kappa$ is the thermal conductivity of the medium.

Heat energy conservation can be expressed in the form

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j}_q = 0, \tag{1.10}$$

where $q$ is the heat energy density (energy per unit volume). It can be expressed as $q = c\rho T$, where $c$ is the specific heat and $\rho$ is the density of the medium.

Combining (1.9) and (1.10) gives the *heat equation*

$$\frac{\partial u}{\partial t} - K\nabla^2 u = 0, \tag{1.11}$$

where $K = \kappa / c\rho$, and $u$ is the temperature. For a stationary temperature distribution, $\partial u/\partial t = 0$, (1.11) becomes the Laplace equation,

$$\nabla^2 u = 0. \tag{1.12}$$

It is also obeyed by the electrostatic potential in the region of space with no charges.

For a uniform rod along the $x$ axis, the heat equation reads

$$u_t - K u_{xx} = 0. \tag{1.13}$$

If heat is produced or absorbed in the rod, this equation takes the form

$$u_t - K u_{xx} = q(x,t), \tag{1.14}$$

where $q(x,t)$ is proportional to the rate of heat production at $x$.

An equation similar to (1.11) describes *diffusion* of a substance in a medium,

$$\frac{\partial u}{\partial t} - D\nabla^2 u = 0, \tag{1.15}$$

where $D$ is the diffusion coefficient, and $u$ is the number density.

Another similar equation is the Schrödinger equation for the wavefunction $\psi(x,y,z,t)$ of a particle in quantum mechanics,

$$i\hbar\frac{\partial \psi}{\partial t} + \left[\frac{\hbar^2}{2m}\nabla^2\psi - U(x,y,z)\psi\right] = 0, \tag{1.16}$$

where $\hbar$ is Planck's constant, $m$ is the mass of the particle, and $U(x,y,z)$ is its potential energy. (1.16) looks like a diffusion equation in imaginary time.

### 1.5 Initial and boundary conditions

1. Wave equation for the string of length $l$.
   (a) Boundary conditions: $u(0,t) = 0$, $u(l,t) = 0$ (fixed ends). The motion of the string will be determined if the initial conditions are specified: $u(x,0) = f(x)$ (displacement), $u_t(x,0) = g(x)$ (velocity).

   (b) If one of the ends of the string is free (e.g., at $x = l$), then $u_x(l,t) = 0$ (the *natural* boundary condition).

   (c) Boundary conditions for a string with a fixed and a driven end: $u(0,t) = 0$, $u(l,t) = h(t)$.

2. Heat equation in one dimension (rod of length $l$).
   (a) Initial condition: $u(x,0) = f(x)$ (temperature at $t = 0$). Boundary conditions: $u(0,t) = T_1$, $u(l,t) = T_2$ (ends are kept at fixed temperature).

   (b) Rod with insulated ends: $u_x(0,t) = 0$, $u_x(l,t) = 0$.

   (c) Newton's law of cooling: the heat flux is proportional to difference between the temperature of the body and ambient temperature. If the temperatures near the two ends of the rod are $T_1$ and $T_2$, the boundary conditions are: $-\kappa u_x(0,t) = \alpha_1[T_1 - u(0,t)]$, $-\kappa u_x(l,t) = \alpha_2[u(l,t) - T_2]$.

### 1.6 Linear differential equations

A linear homogeneous differential expression for a function $u(x,y,\dots)$ is

$$L[u] = Au + Bu_x + Cu_y + \dots + Du_{xx} + Eu_{xy} + \dots, \tag{1.17}$$

where

$$L = A + B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y} + \dots + D\frac{\partial^2}{\partial x^2} + E\frac{\partial^2}{\partial x \partial y} + \dots. \tag{1.18}$$

is a linear differential operator, for which

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]. \tag{1.19}$$

The general linear differential equation has the form

$$L[u] = f(x,y,\dots). \tag{1.20}$$

If $f = 0$, the equation is homogeneous, $f \neq 0$ – inhomogeneous.

Superposition principle: if $u_1$, $u_2$, ... are solutions of the homogeneous equation, then $c_1u_1 + c_2u_2 + \dots$ is also a solution.

If $u_p$ is a solution of (1.20) with $f \neq 0$, other solutions of the inhomogeneous equation can be constructed as $u = u_p + c_1u_1 + c_2u_2 + \dots$.

Boundary conditions can also be homogeneous, e.g., $u(0,t) = 0$, $u_x(0,t) = 0$, or $\alpha u(0,t) + \beta u_x(0,t) = 0$, or inhomogeneous: $u(0,t) = T$, $u(l,t) = h(t)$, etc.

# 2 The method of variable separation

## 2.1 Variable separation in partial differential equations

Particular solutions of a linear homogeneous equation,

$$L[u] = 0, \tag{2.1}$$

in $n$ independent variables $x,\ y, \ldots,$ can sometimes be found in the form

$$u(x, y, \ldots) = X(x)Y(y)\ldots\ . \tag{2.2}$$

Variable separation involves substituting (2.2) into (2.1), introducing $n-1$ *separation constants* $k_1,\ k_2,\ \ldots,\ k_{n-1}$, and solving $n$ ordinary differential equations for $X(x)$, $Y(y)$, $\ldots$.

These solutions can be combined using the superposition principle to form a more general family of solutions (subject to certain boundary conditions).

## 2.2 Variable separation for the wave equation for a string

We seek solution of the wave equation for a string of length $l$ with fixed ends, $u(0,t) = u(l,t) = 0$, in the form

$$u(x,t) = v(x)q(t). \tag{2.3}$$

Substitution into (1.5) yields

$$\underbrace{\frac{1}{q(t)}\frac{d^2q}{dt^2}}_{\lambda} = \underbrace{c^2\frac{1}{v(x)}\frac{d^2v}{dx^2}}_{\lambda}, \tag{2.4}$$

and each side must be equal to a constant. Assuming that the separation constant is negative, $\lambda = -\omega^2$, we have

$$q'' + \omega^2 q = 0, \tag{2.5}$$

whose general solution is

$$q(t) = A\cos(\omega t + \alpha). \tag{2.6}$$

It describes harmonic oscillations with frequency $\omega$, while $A$ and $\alpha$ are arbitrary constants. The general solution of the coordinate equation,

$$v'' + k^2 v = 0, \tag{2.7}$$

where $k = \omega/c$ is the *wavenumber*, is

$$v(x) = C_1\cos kx + C_2\sin kx. \tag{2.8}$$

Imposing the boundary conditions, $v(0) = v(l) = 0$, we have $C_1 = 0$ and

$$k = \frac{\pi n}{l}, \quad n = 1, 2, \ldots \tag{2.9}$$

Hence, the wave equation has the solutions

$$u(x,t) = C \sin \frac{n\pi x}{l} \cos(\omega_n t + \alpha), \tag{2.10}$$

where $\omega_n = n\pi c/l$. Such solutions are known as *standing waves*, and are also called *eigenvibrations*, or *modes*.

## 2.3 Variable separation for the circular membrane. Bessel functions

The vibrations of a membrane are described by the two-dimensional wave equation (1.7). For a circular membrane we use plane polar coordinates $r$ and $\varphi$, and $u(r, \varphi, t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right] = 0, \tag{2.11}$$

with the boundary condition $u(a, \varphi, t) = 0$ (fixed edge), where $a$ is the radius of the membrane.

Using variable separation, we seek solution in the form

$$u(r, \varphi, t) = v(r, \varphi) q(t). \tag{2.12}$$

Substitution into (2.11) gives (2.5) for $q(t)$, with solution (2.6), and

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} + k^2 v = 0, \tag{2.13}$$

where $k = \omega/c$, for the spatial part $v(r, \varphi)$. Separating variables further,

$$v(r, \varphi) = R(r) \Phi(\varphi), \tag{2.14}$$

and denoting the separation constant by $-m^2$, we obtain

$$\Phi'' + m^2 \Phi = 0, \tag{2.15}$$

with the general solution

$$\Phi(\varphi) = A \cos m\varphi + B \sin m\varphi. \tag{2.16}$$

Since $\Phi(\varphi)$ is periodic with period $2\pi$, we must have $m = 0, 1, 2, \ldots$.

The radial part satisfies

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( k^2 - \frac{m^2}{r^2} \right) R = 0. \tag{2.17}$$

Introducing $z = kr$, we obtain for $R(z)$:

$$\frac{d^2R}{dz^2} + \frac{1}{z}\frac{dR}{dz} + \left(1 - \frac{m^2}{z^2}\right) R = 0, \tag{2.18}$$

or

$$z^2 R'' + zR' + (z^2 - m^2)R = 0. \tag{2.19}$$

This is the *Bessel equation*. We seek its solution in the form $R = \sum_{n=0}^{\infty} a_n z^n$, which leads to the recurrence relation,

$$a_n = -\frac{a_{n-2}}{(n-m)(n+m)}. \tag{2.20}$$

Setting $a_m = 1/(2^m m!)$ (and $a_n = 0$ for $n < m$), we obtain $R = J_m(z)$, where

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{k!(m+k!)} \tag{2.21}$$

is the *Bessel function* of order $m$. These functions are finite at $z = 0$ ($J_0(0) = 1$, and $J_m(0) = 0$ for $m > 0$). For $z > 0$ they oscillate, and each Bessel function has infinitely many roots $z_{m,n}$, $J_m(z_{m,n}) = 0$, $n = 1, 2, \ldots$.

Hence, the solution of the radial equation (2.17) finite at $r = 0$ is $R(r) = J_m(kr)$. To satisfy the boundary condition $R(a) = 0$, we require $ka = z_{m,n}$, which gives

$$k = \frac{z_{m,n}}{a} \equiv k_{m,n}. \tag{2.22}$$

The corresponding frequencies are

$$\omega = \frac{z_{m,n}c}{a} \equiv \omega_{m,n}. \tag{2.23}$$

Combining the radial, angular and temporal parts of the solution, we have

$$u(r, \varphi, t) = J_m(k_{m,n}r)(A\cos m\varphi + B\sin m\varphi)\cos(\omega_{m,n}t + \alpha). \tag{2.24}$$

## 2.4 Variable separation for the heat equation

Consider the heat equation in one dimension for $u(x,t)$,

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.25}$$

for a rod of length $l$ ($0 \le x \le l$), whose ends are kept at zero temperature,

$$u(0,t) = 0, \quad u(l,t) = 0. \tag{2.26}$$

Seeking solution in the form

$$u(x,t) = v(x)q(t), \tag{2.27}$$

we substitute (2.27) into (2.25) and obtain:

$$v(x)\frac{\partial q}{\partial t} - q(t)K\frac{\partial^2 v}{\partial x^2} = 0. \tag{2.28}$$

The partial derivatives in (2.28) can be replaced by the total ones (i.e., $\partial$ by $d$), since $q$ depends only on $t$, and $v$ depends only on $x$. Dividing (2.28) through by $v(x)q(t)$ yields

$$\underbrace{\frac{1}{q(t)}\frac{dq}{dt}}_{\text{const}} - \underbrace{\frac{K}{v(x)}\frac{d^2 v}{dx^2}}_{\text{const}} = 0. \tag{2.29}$$

In this equation the first term depends only on $t$, while the second one only on $x$. For the equation to be valid <u>for all</u> $x$ and $t$, the two terms must be equal to a constant. Denoting the separation constant by $-\lambda$ (with $\lambda > 0$ as we shall see below), we have for the first term in (2.29):

$$\frac{1}{q(t)}\frac{dq}{dt} = -\lambda,$$

or

$$\frac{dq}{dt} = -\lambda q(t). \tag{2.30}$$

Solving this equation,

$$\int \frac{dq}{q} = -\int \lambda dt,$$
$$\ln q = -\lambda t + C,$$
$$q = e^C e^{-\lambda t},$$

and replacing $e^C$ by $C$, where $C$ is an arbitrary constant, we obtain

$$q(t) = Ce^{-\lambda t}. \tag{2.31}$$

Note that for $\lambda > 0$ this is a decreasing function of time, while for $\lambda < 0$ , $q(t) \to \infty$ for $t \to \infty$. Hence, assuming that $-\lambda$ is positive, would lead to an unrealistic <u>heating up</u> of the rod. This is a physical justification of our choice of the sign of the separation constant.

The $x$-coordinate part of (2.29) is

$$\frac{K}{v(x)}\frac{d^2 v}{dx^2} = -\lambda,$$

or

$$\frac{d^2 v}{dx^2} + \frac{\lambda}{K}v = 0. \tag{2.32}$$

This equation has the general solution,

$$v(x) = A\cos\left(\sqrt{\frac{\lambda}{K}}x\right) + B\sin\left(\sqrt{\frac{\lambda}{K}}x\right), \tag{2.33}$$

where $A$ and $B$ are arbitrary constants. To satisfy the boundary conditions (2.26) for $u(x,t)$ in the form (2.27), we require

$$v(0) = 0, \quad v(l) = 0. \tag{2.34}$$

Applying the first of these to (2.33) gives $A = 0$, and the second then reads

$$B \sin\left(\sqrt{\frac{\lambda}{K}}\, l\right) = 0.$$

For a nonzero $v(x)$, the sine function above must be zero, which gives

$$\sqrt{\lambda/K}\, l = n\pi, \qquad n = 1,\, 2, \ldots$$

This determines the possible values of the separation constant[3]

$$\lambda = \frac{n^2\pi^2}{l^2} K, \tag{2.35}$$

and the corresponding solutions

$$v(x) = B \sin\frac{n\pi x}{l}. \tag{2.36}$$

Combining Eqs. (2.31) and (2.36), and using (2.35), we obtain

$$u(x,t) = B_n \sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2/l^2)Kt}, \tag{2.37}$$

where $B_n$ is a new arbitrary constant, and $n = 1,\, 2, \ldots$.

By the *superposition principle*, one can obtain a more general solution of equation (2.25) with boundary conditions (2.26), as a superposition (i.e., linear combination) of solutions (2.37) with different $n$:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2/l^2)Kt}$$

The coefficients $B_n$ can be determined from the initial condition, e.g.,

$$u(x,0) = f(x), \tag{2.38}$$

where $f(x)$ describes the temperature distribution in the rod at $t = 0$. This gives

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x). \tag{2.39}$$

To find $B_n$, we can multiply (2.39) by $\sin(m\pi x/l)$ and integrate term-by-term from 0 to $l$ using

$$\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx = \frac{l}{2}\delta_{nm}, \quad m, n = 1,\, 2, \ldots \tag{2.40}$$

[3]For $\lambda < 0$, the general solution of (2.32) is $v(x) = Ae^{\sqrt{-\lambda/K}x} + Be^{-\sqrt{-\lambda/K}x}$, and the boundary conditions (2.34) cannot be satisfied for nonzero $A$ or $B$.

where $\delta_{mn}$ is the Kronecker delta symbol (3.14).[4] As a result, only the term with $n = m$ gives a nonzero contribution on the left-hand side, yielding

$$B_m = \frac{2}{l} \int_0^l f(x) \sin \frac{m\pi x}{l}\, dx, \tag{2.41}$$

The series on the left-hand side of (2.39) is a *Fourier series.* We study Fourier series in Chapter 3. In particular, one needs to verify that substitution of (2.41) into (2.39) gives an identity for any "good" function $f(x)$.

[4]To verify (2.40) and (3.4)–(3.6), one can use $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$, $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$, and $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$.

# 3 Fourier series

## 3.1 Fourier expansion of a function

A function $f(x)$ defined on $(-\pi, \pi)$ can be represented by the *Fourier series*,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{3.1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{3.2}$$

Equations (3.2) can be obtained if we multiply

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{3.3}$$

by either $\cos mx$ $(m = 0, 1, \dots)$ or $\sin mx$ $(n = m, 2, \dots)$, and integrate between $-\pi$ and $\pi$, using the orthogonality relations,[4]

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 2\pi, & m = n = 0, \\ \pi, & m = n > 0, \\ 0, & m \neq n, \end{cases} \tag{3.4}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} \pi, & m = n, \\ 0, & m \neq n, \end{cases} \tag{3.5}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0. \tag{3.6}$$

Convergence Theorem. Let $f$ be a piecewise smooth function[5] on $(-\pi, \pi)$, and let all its discontinuities be finite "jumps". Then the Fourier series (3.1) converges to $f(x)$ for all $x$ where $f$ is continuous. If $f$ has a discontinuity at $x = \xi$, then the Fourier series converges to $\frac{1}{2}[f(\xi - 0) + f(\xi + 0)]$.[6]

The Fourier series is periodic with period $2\pi$. It provides a *periodic extension* of $f(x)$ onto the whole real axis. At $x = -\pi$ and $x = \pi$ the Fourier series equals $\frac{1}{2}[f(\pi - 0) + f(\pi + 0)]$.

The Fourier series for an even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{3.7}$$

[5] i.e., its derivative is piecewise continuous.

[6] $f(\xi - 0) = \lim_{x\to\xi+0} f(x)$ and $f(\xi - 0) = \lim_{x\to\xi-0} f(x)$ are the "right-hand" and "left-hand" limits of $f(x)$ at $x = \xi$.

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx. \tag{3.8}$$

The Fourier series for an odd function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{3.9}$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx. \tag{3.10}$$

For a function defined on $0 < x < \pi$, equations (3.7) and (3.9) provide an even and odd periodic extensions onto the whole real axis.

The Fourier series can be written in the complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{3.11}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x) dx. \tag{3.12}$$

To verify this, multiply (3.11) by $e^{-imx}$ and integrate from $-\pi$ to $\pi$, using

$$\int_{-\pi}^{\pi} e^{-imx} e^{inx} dx = 2\pi\delta_{mn}, \tag{3.13}$$

where

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases} \tag{3.14}$$

is the *Kronecker* delta symbol.

Examples.

1. The Fourier series for

$$f(x) = \begin{cases} \frac{\pi - x}{2}, & 0 < x < \pi, \\ -\frac{\pi - x}{2}, & -\pi < x < 0, \end{cases}$$

is

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

2. The Fourier series for $f(x) = e^x$ on $-\pi < x < \pi$ is

$$\frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{1+n^2}\cos nx - \frac{n}{1+n^2}\sin nx\right).$$

Parseval's relation. Consider a piecewise smooth function $f(x)$ on $(-\pi, \pi)$ with a convergent Fourier series (3.3). Squaring both sides of this equation and integrating between $-\pi$ and $\pi$, we have

$$\int_{-\pi}^{\pi} f^2(x)dx = \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx)\right] \times \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right] dx.$$

Here we have written the square on the right-hand side as a product of two identical Fourier sums with different summation indices, $m$ and $n$, to avoid confusion.

Expanding the product of the two square brackets on the right-hand side and changing the order of summation and integration,[7] we obtain

$$\int_{-\pi}^{\pi} \left(\frac{a_0}{2}\right)^2 dx + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} a_m a_n \cos mx \cos nx \, dx + \int_{-\pi}^{\pi} b_m b_n \sin mx \sin nx \, dx\right] + \ldots \qquad (3.15)$$

where $\ldots$ stands for all the other contributions, such as $\frac{a_0}{2}\int_{-\pi}^{\pi} a_n \cos nx \, dx$, $\int_{-\pi}^{\pi} a_m b_n \cos mx \sin nx \, dx$, etc. Due to orthogonality relations (3.4)–(3.6), these contributions vanish, and only the first term and those with $m = n$ in the double sum in (3.15) are nonzero. Hence,

$$\int_{-\pi}^{\pi} f^2(x)dx = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right], \qquad (3.16)$$

which is known as *Parseval's relation.*

It has a simple geometric interpretation. The Fourier expansion of a function can be compared to the expansion of a vector $\mathbf{a}$ in an orthonormal basis $\mathbf{e}_n$ $(n = 1, \ldots, N)$, $\mathbf{a} = \sum_{n=1}^{N} a_n \mathbf{e}_n$, where $a_n$ are the components of $\mathbf{a}$, and $\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{nm}$. The square of $\mathbf{a}$ is then given by $\mathbf{a} \cdot \mathbf{a} = \sum_{n=1}^{N} a_n^2$, which is a finite-dimension vector analogue of Parseval's relation (3.16).

Example. The Fourier series for $f(x) = x^2$ on $(-\pi, \pi)$ is:

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2},$$

[7] This requires uniform convergence of the Fourier series, which is guaranteed for a piecewise smooth function on every closed interval on which the function is continuous, R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1 (Interscience Publishers, New York, 1953) Ch. II, § 5.

i.e., $a_0 = 2\pi^2/3$ and $a_n = (-1)^n/n^2$ (see Problem sheet 3). Given that $\int_{-\pi}^{\pi} x^4 dx = 2\pi^5/5$, we have from (3.16):

$$\frac{2\pi^5}{5} = \pi\left[\frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}\right],$$

which yields

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Gibbs's phenomenon. According to the convergence theorem, partial sums of the Fourier series for a piecewise smooth function approach the graph of $f(x)$ in every closed interval that does not contain the discontinuity, as the number of terms increases.

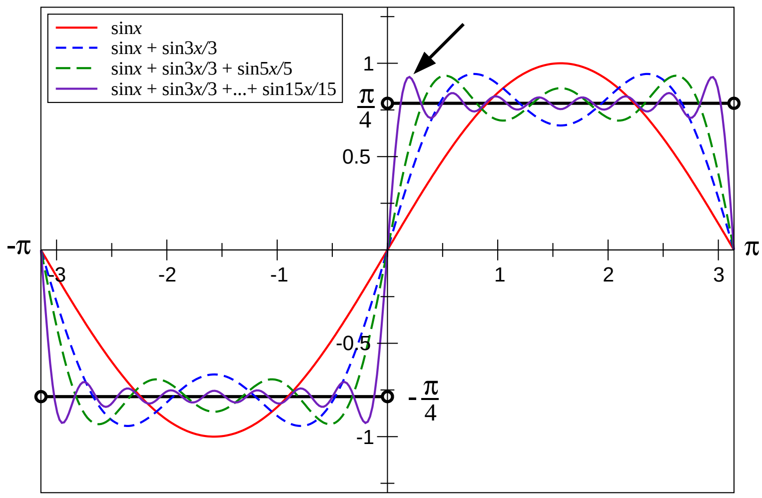
We illustrate this for

$$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0, \\ \frac{\pi}{4}, & 0 < x < \pi. \end{cases} \tag{3.17}$$

which has the Fourier series

$$\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1}. \tag{3.18}$$

The figure below shows $f(x)$ together with the partial sums of its Fourier series containing 1, 2, 3, and 8 terms.




At the discontinuity the partial sums display oscillations which become progressively narrower and move closer to the discontinuity. However, they always "overshoot" (e.g., see the maximum nearest to the discontinuity shown by arrow on the graph), and the total oscillation in the approximating curve does not approach the jump in $f(x)$. This is known at the *Gibbs phenomenon.*

To analyse this effect, consider the partial sum of first $N+1$ terms of the Fourier series (3.18),

$$S_N(x) = \sum_{m=0}^{N} \frac{\sin(2m+1)x}{2m+1}. \tag{3.19}$$

Its derivative, $S_N'(x) = \sum_{m=0}^{N} \cos(2m+1)x$, is equal to the real part of the geometric series,

$$\sum_{m=0}^{N} e^{i(2m+1)x} = e^{ix}\,\frac{1-e^{i2(N+1)x}}{1-e^{i2x}} = \frac{1-e^{i2(N+1)x}}{e^{-ix}-e^{ix}} = \frac{e^{i2(N+1)x}-1}{2i\sin x},$$

so that

$$S_N'(x) = \mathrm{Re}\,\frac{e^{i2(N+1)x}-1}{2i\sin x} = \frac{\sin 2(N+1)x}{2\sin x}. \tag{3.20}$$

Let $x = x_m$ be the first maximum of $S_N(x)$ at $x > 0$. From $S_N'(x_m) = 0$, we find $x_m = \pi/[2(N+1)]$. Given that $S_N(0) = 0$, we can find the value of the partial sum at this maximum, $S_N(x_m)$, from the integral,

$$S_N(x_m) = \int_0^{x_m} S_N'(x)dx = \int_0^{x_m} \frac{\sin 2(N+1)x}{2\sin x}\,dx \simeq \int_0^{x_m} \frac{\sin 2(N+1)x}{2x}\,dx.$$

In the last expression we used $\sin x \simeq x$, which is valid for $x \ll 1$, and can be used for large $N$ since $x_m \ll 1$. Using variable substitution $t = 2(N+1)x$ in the last integral, we obtain the partial sum at the maximum nearest to the discontinuity as

$$S_N(x_m) = \frac{1}{2}\int_0^{\pi} \frac{\sin t}{t}\,dt \approx 0.9260, \tag{3.21}$$

where the last value was obtained numerically. It is about 18% higher than $f(0+0) = \pi/4 \approx 0.7854$. This means that near the discontinuity the partial sums overshoot the graph of $f(x)$ by about 9% of the total size of the jump.

Fourier series on an arbitrary symmetric interval. For $f(x)$ defined on $[-l, l]$, the Fourier series can be obtained from (3.1) and (3.2) by a variable change, $x \to \pi x/l$. This gives

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \tag{3.22}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{3.23}$$

Similarly, one obtains half-range Fourier cosine series for even functions $f(x)$, for which $b_n = 0$ and

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \tag{3.24}$$

and sine series for odd $f(x)$, for which $a_n = 0$ and

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{3.25}$$

## 3.2 Application of Fourier's method to PDE

1. Vibrations of a string. Solve the wave equation (1.5) for the string of length $l$ with boundary conditions,

$$u(0,t) = 0, \quad u(l,t) = 0,$$

(fixed ends), and initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0,$$

for

$$f(x) = \begin{cases} h\frac{x}{a}, & 0 \le x \le a, \\ h\frac{l-x}{l-a}, & a \le x \le l, \end{cases}$$

(string stretched to $u = h$ at $x = a$, and released from rest).

Answer:

$$u(x,t) = \frac{2hl^2}{\pi^2 a(l-a)} \sum_{n=1}^{\infty} \frac{\sin \dfrac{n\pi a}{l}}{n^2} \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t.$$

2. Heat equation in one dimension. Solve the heat equation (1.13) for a rod of length $l$, whose ends are kept at zero temperature,

$$u(0,t) = 0, \quad u(l,t) = 0,$$

given the initial temperature distribution

$$u(x,0) = f(x).$$

Answer:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-(n^2\pi^2/l^2)Kt},$$

where

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

3. Laplace's equation for a disk. Solve Laplace's equation (1.12) in two dimensions for a disk of radius $a$ using plane polar coordinates, for $u(r,\varphi)$ whose values are fixed at the boundary,

$$u(a,\varphi) = f(\varphi). \tag{3.26}$$

Laplace's equation in plane polar coordinates reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{3.27}$$

Using variable separation, one finds the solution finite at the origin as

$$u(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi), \tag{3.28}$$

(see Examples in Problem sheet 2). The coefficients $a_n$ and $b_n$ are found from (3.26) using the Fourier formulae. Substituting them into (3.28) gives

$$u(r,\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\psi - \varphi) \right] f(\psi) d\psi. \tag{3.29}$$

The expression in square brackets is found as the real part of a complex geometric series. This gives the answer in the form

$$u(r,\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(a^2 - r^2) f(\psi) d\psi}{a^2 - 2ar\cos(\psi - \varphi) + r^2}, \tag{3.30}$$

known as *Poisson's integral.*[8]

In particular, for $u$ at the origin, $r = 0$, we have

$$u(0,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) d\psi, \tag{3.31}$$

which shows that $u$ at the centre is equal to its average over the circle. Hence, the solutions of Laplace's equation (known as *harmonic functions*) in a domain cannot have minima or maxima inside the domain.[9]

For $r = a$ the left-hand side of the Poisson integral (3.30) must be equal to $f(\varphi)$, according to (3.26). This means that for $r \to a - 0$,

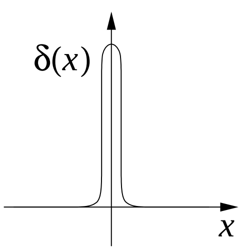
$$\frac{1}{2\pi} \frac{(a^2 - r^2)}{a^2 - 2ar\cos(\psi - \varphi) + r^2} \longrightarrow \delta(\psi - \varphi), \tag{3.32}$$

where $\delta(\psi - \varphi)$ is a special function for which

$$\int_{-\pi}^{\pi} \delta(\psi - \varphi) f(\psi) d\psi = f(\varphi). \tag{3.33}$$

We see that of all the values of $\psi$ in the above integral, only the point $\psi = \varphi$ contributes. Hence, we have to conclude that $\delta(\psi - \varphi) = 0$ for $\psi \neq \varphi$. On the other hand, using $f = 1$ in (3.33), we will have

$$\int \delta(\psi - \varphi) d\psi = 1, \tag{3.34}$$




where the integral is over *any* interval containing point $\varphi$. The function that has these properties is known as the *Dirac delta function.* One can think of $\delta(x)$ as an infinitely narrow and infinitely tall spike at $x = 0$, such that the area under the graph equals unity.

[8]Note that the denominator is the squared distance between points $(r,\varphi)$ and $(a,\psi)$.

[9]Otherwise, $u$ at this point will be either smaller or greater than all of its values on a small circle surrounding the extremum, which would contradict (3.31).

# 4 Integral transform methods: Fourier and Laplace

## 4.1 Fourier integral

The Fourier integral allows one to represent $f(x)$ on the whole real axis.

Consider the Fourier series for $f(x)$ on $-l \leq x \leq l$ in complex form,[10]

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}, \tag{4.1}$$

$$c_n = \frac{1}{2l} \int_{-l}^{l} e^{-in\pi x/l} f(x) dx, \tag{4.2}$$

Taken together, these equation give

$$f(x) = \frac{1}{2l} \sum_{n=-\infty}^{\infty} e^{in\pi x/l} \int_{-l}^{l} e^{-in\pi\xi/l} f(\xi) d\xi. \tag{4.3}$$

Introducing $p_n = n\pi/l$ and $\Delta p = p_{n+1} - p_n = \pi/l$, we re-write (4.3) as

$$f(x) = \frac{1}{2\pi} \sum_{p_n=-\infty}^{\infty} e^{ip_n x} \Delta p \int_{-l}^{l} e^{-ip_n \xi} f(\xi) d\xi. \tag{4.4}$$

The sum in (4.4) is a Riemann sum. In the limit $l \to \infty$, $\Delta p \to 0$, it becomes an integral over $p$, and we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} dp \int_{-\infty}^{\infty} e^{-ip\xi} f(\xi) d\xi. \tag{4.5}$$

Commonly, this relation is written with $p$ replaced by $-p$, as a combination of two formulae,

$$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) dx, \tag{4.6}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} F(p) dp. \tag{4.7}$$

Here $F(p) \equiv \mathcal{F}[f]$ is the *Fourier transform* of $f(x)$, and $f(x) \equiv \mathcal{F}^{-1}[F]$ is the inverse Fourier transform of $F(p)$.

To verify these relations, we substitute (4.6) (with $x$ changed to $\xi$, to avoid confusion) into (4.7), and make the integration limits in (4.7) finite, e.g., $-P$ and $P$,

$$\frac{1}{2\pi} \int_{-P}^{P} e^{-ipx} dp \int_{-\infty}^{\infty} e^{ip\xi} f(\xi) d\xi, \tag{4.8}$$

[10] Compare (3.1), (3.2) and (3.22), (3.23) with (3.11), (3.12).

and will later take the limit $P \to \infty$. Changing the order of integration in (4.8), we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-P}^{P} e^{ip(\xi-x)}dp = \int_{-\infty}^{\infty} f(\xi)\frac{\sin P(\xi-x)}{\pi(\xi-x)}d\xi. \tag{4.9}$$

The function

$$\frac{\sin P(\xi-x)}{\pi(\xi-x)} \tag{4.10}$$

in the integrand is even, and has a maximum at $\xi \to x$, where it equals $P/\pi$. The "width" of this maximum is $|\xi - x| \sim \pi/P$. One can also show that

$$\int_0^{\infty} \frac{\sin Ps}{s}ds = \int_0^{\infty}\frac{\sin s}{s}ds = \frac{\pi}{2}, \tag{4.11}$$

which is known as the *Dirichlet integral.* This means that in the limit $P \to \infty$, (4.10) becomes the $\delta$-function,

$$\lim_{P\to\infty}\frac{\sin P(\xi-x)}{\pi(\xi-x)} = \delta(\xi - x). \tag{4.12}$$

Hence, taking the limit $P \to \infty$ of the right-hand side of (4.9), we obtain

$$\int_{-\infty}^{\infty} f(\xi)\delta(\xi-x)d\xi = f(x),$$

as required. Formally, this result proves the following identity:

$$\int_{-\infty}^{\infty} e^{ip(\xi-x)}dp = 2\pi\delta(\xi-x). \tag{4.13}$$

Examples. By direct calculation we show:

1. $\mathcal{F}\left[e^{-\alpha|x|}\right] = \dfrac{1}{\sqrt{2\pi}}\dfrac{\alpha}{\alpha^2+p^2}, \quad \mathcal{F}^{-1}\left[\dfrac{1}{\sqrt{2\pi}}\dfrac{\alpha}{\alpha^2+p^2}\right] = e^{-\alpha|x|}.$
2. $\mathcal{F}\left[e^{-x^2/2}\right] = e^{-p^2/2}, \quad \mathcal{F}^{-1}\left[e^{-p^2/2}\right] = e^{-x^2/2}.$

To prove the latter, we use (and derive) the following useful formula:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}dx = \sqrt{\frac{\pi}{\alpha}}e^{\beta^2/4\alpha}. \tag{4.14}$$

Fourier cosine and sine transforms. Assuming that $f(x)$ is even one obtains the Fourier cosine transform formulae,

$$F_c(p) \equiv \mathcal{F}_c[f] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos px\, dx, \tag{4.15}$$

$$f(x) = \mathcal{F}_c^{-1}[F_c] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(p)\cos px\, dp. \tag{4.16}$$

For an odd function one obtains the Fourier sine transform equations,

$$F_s(p) \equiv \mathcal{F}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx, \tag{4.17}$$

$$f(x) = \mathcal{F}_s^{-1}[F_s] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(p) \sin px \, dp. \tag{4.18}$$

Both of these can be used to transform functions defined on $0 \leq x < \infty$.

## 4.2 Application of Fourier transforms to PDE

Fourier transforms can be used to solve PDE on infinite or semi-infinite intervals.

Problem 1. Solve the heat equation, $u_t - K u_{xx} = 0$ on $-\infty < x < \infty$ with the initial condition $u(x, 0) = f(x)$.

Answer:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(\xi - x)^2/4Kt}}{\sqrt{4\pi K t}} f(\xi) d\xi.$$

Note that for $t \to 0$ the left-hand side becomes $f(x)$, which means that

$$\lim_{t \to 0} \frac{e^{-(\xi - x)^2/4Kt}}{\sqrt{4\pi K t}} = \delta(\xi - x),$$

This can be proved directly by examining the behaviour of this function for $t \to 0$, and proving that its integral equals unity.

Problem 2. Solve the one-dimensional wave equation, $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty$ with the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.

Answer:

$$u(x,t) = \frac{1}{2}[f(x - ct)] + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi.$$

In particular, for $g(x) = 0$, the solution describes how the initial wave shape $f(x)$ breaks into two waves, $\frac{1}{2} f(x - ct)$ and $\frac{1}{2} f(x + ct)$, travelling in the opposite directions.

## 4.3 Laplace transform

The Laplace transform of $f(t)$ is

$$F(p) \equiv \mathcal{L}[f] = \int_0^\infty f(t) e^{-pt} dt, \tag{4.19}$$

where $f(t)$ is assumed to be

1. piecewise smooth on $0 \leq t < \infty$,

2. $f(t) = 0$ for $t < 0$,
3. For $t \to \infty$, $|f(t)| < Me^{at}$, for some $M$ and $a$.

The last condition ensures that the integral in (4.19) converges for $p > a$ (or $\mathrm{Re}\, p > a$, if we consider complex $p$).

The inverse Laplace transform is an integral in the complex $p$ plane along the straight line parallel to the imaginary axis (*Mellin's formula*),

$$f(t) = \mathcal{L}^{-1}[F] = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(p) e^{pt} dp, \tag{4.20}$$

where the integration path is chosen so that $F(p)$ is regular for $\mathrm{Re}\, p > \sigma$.

Example: $\mathcal{L}\left[e^{\alpha t}\right] = \dfrac{1}{p - \alpha}$, $\quad \mathcal{L}^{-1}\left[\dfrac{1}{p-\alpha}\right] = e^{\alpha t}$ for $t \geq 0$, and 0 for $t < 0$.

Table of Laplace transforms

| Original | Transform | Comment |
|---|---|---|
| $f(t)$ | $F(p)$ | |
| $Cf(t)$ | $CF(p)$ | These two relations mean |
| $f(t) + g(t)$ | $F(p) + G(p)$ | that $\mathcal{L}$ is a linear operator |
| $f(\alpha t)$ | $\frac{1}{\alpha} F\left(\frac{p}{\alpha}\right)$ | |
| $f'(t)$ | $pF(p) - f(0)$ | |
| $f''(t)$ | $p^2 F(p) - pf(0) - f'(0)$ | |
| $\Theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ | $\frac{1}{p}$ | Heaviside step function |
| $e^{\alpha t}$ | $\frac{1}{p - \alpha}$ | |
| $t^n e^{-\alpha t}$ | $\frac{n!}{(p-\alpha)^{n+1}}$ | Use $\int_0^\infty t^n e^{\alpha t} e^{-pt} dt$ $= \left(\frac{\partial}{\partial \alpha}\right)^n \int_0^\infty e^{\alpha t} e^{-pt} dt$ |
| $\sin \omega t$ | $\frac{\omega}{p^2 + \omega^2}$ | |
| $\cos \omega t$ | $\frac{p}{p^2 + \omega^2}$ | |

Shift theorems.

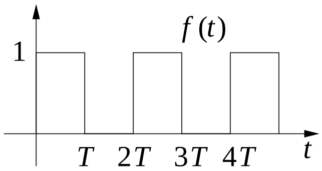
$$\mathcal{L}[e^{\alpha t} f(t)] = F(p - \alpha), \tag{4.21}$$

$$\mathcal{L}[f(t-a)] = e^{-pa} F(p). \tag{4.22}$$

Examples.

1. $\mathcal{L}\left[e^{\alpha t}\sin\omega t\right] = \dfrac{\omega}{(p-\alpha)^2+\omega^2}$.

2. $\mathcal{L}[\Theta(t-a)] = \dfrac{e^{-pa}}{p}$.




3. For the function $f(t) = \sum_{n=0}^{\infty}(-1)^n\Theta(t-nT)$,

$$F(p) = \frac{1}{p(1-e^{-pT})}.$$

Convolution theorem.

$$\mathcal{L}\left[\int_0^t g(t-\tau)f(\tau)d\tau\right] = F(p)G(p), \tag{4.23}$$

where $G(p) = \mathcal{L}[g]$, and the quantity in brackets is the *convolution* of functions $f$ and $g$.

Example. Find the original of the Laplace transform $F(p) = \dfrac{1}{(p-\alpha)(p-\beta)}$.

Using partial fractions, $\frac{1}{(p-\alpha)(p-\beta)} = \frac{1}{\alpha-\beta}\left[\frac{1}{p-\alpha} - \frac{1}{p-\alpha}\right]$, we obtain

$$\mathcal{L}^{-1}[F] = \frac{1}{\alpha-\beta}\left[e^{\alpha t} - e^{\beta t}\right].$$

Alternatively, $F(p)$ is a product of two Laplace transforms, so using the convolution theorem,

$$\mathcal{L}^{-1}[F] = \int_0^t e^{\alpha\tau}e^{\beta(t-\tau)}d\tau = \frac{1}{\alpha-\beta}\left[e^{\alpha t} - e^{\beta t}\right].$$

## 4.4 Applications of Laplace transforms to ordinary and partial DE

Problem 1. Solve the 2nd-order inhomogeneous linear ODE with constant coefficients for $y(t)$:

$$ay'' + by' + cy = f(t).$$

Answer:

$$\begin{aligned} y(t) = {} & \frac{y'(0)-\alpha_2 y(0)}{\alpha_1-\alpha_2}e^{\alpha_1 t} + \frac{y'(0)-\alpha_1 y(0)}{\alpha_2-\alpha_1}e^{\alpha_2 t} \\ & + \frac{1}{\alpha_1-\alpha_2}\int_0^t f(\tau)\left[e^{\alpha_1(t-\tau)} - e^{\alpha_2(t-\tau)}\right]d\tau. \end{aligned} \tag{4.24}$$

where $\alpha_1$ and $\alpha_2$ are the roots of the auxiliary equation $a\alpha^2 + b\alpha + c = 0$.

Example. The forced vibrations of a harmonic oscillator acted upon by an arbitrary force, $y'' + \omega^2 y = f(t)$, are given by the particular integral from (4.24) as

$$y_p(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau.$$

Problem 2. Solve the wave equation for a string of length $l$, initially at rest, $u(x, 0) = 0$, $u_t(x, 0) = 0$, whose right end is fixed, $u(l, t) = 0$, and left end is driven in a given way, $u(0, t) = f(t)$.

Answer:

$$u(x,t) = \sum_{n=0}^{\infty} \left[ f\left(t - \frac{x}{c} - \frac{2nl}{c}\right) - f\left(t + \frac{x}{c} - \frac{2(n+1)l}{c}\right)\right],$$

is a superposition of waves travelling to the right (first term) after $n$ reflections at $x = 0$, and to the left (second term) after $n + 1$ reflections at $x = l$.

# 5 Orthogonal expansions. Sturm-Liouville problem

## 5.1 Inner product and norm. Orthogonal systems of functions

Definition: the *inner product* of two functions, $f$ and $g$, piecewise smooth on $[a, b]$, is

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{5.1}$$

The inner product is symmetric, $(f, g) = (g, f)$, and *bilinear*,

$$(f_1 + f_2, g) = (f_1, g) + (f_2, g), \tag{5.2}$$

$$(Cf, g) = C(f, g), \tag{5.3}$$

and the same for $g$. Hence, the inner product is distributive.[11]

Definition: the *norm* of a function $f$ is $\|f\| \geq 0$,

$$\|f\|^2 = (f, f). \tag{5.4}$$

A function $f$ is called *normalised* if $\|f\| = 1$.

Schwartz's inequality:

$$(f, g)^2 \leq \|f\|^2\|g\|^2. \tag{5.5}$$

Proof: consider $(\lambda f + g, \lambda f + g) \geq 0$, and use the fact that the discriminant of the quadratic expression in $\lambda$ is not positive.

Definition: functions $f$ and $g$ are *orthogonal* if

$$(f, g) = 0. \tag{5.6}$$

Definition: a system of functions $\varphi_1$, $\varphi_2$, ..., is called *orthonormal* if

$$(\varphi_i, \varphi_j) = \delta_{ij}. \tag{5.7}$$

Example of an orthonormal system on $[0, 2\pi]$:

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \frac{\cos 2x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \ldots .$$

[11] The inner product of two functions plays a role similar to the scalar product of two vectors, $\mathbf{a}\cdot\mathbf{b}$. The norm is the analogue of the length of a vector, whose square is $a^2 = \mathbf{a}\cdot\mathbf{a}$. Since $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, Schwartz's inequality for vectors, $(\mathbf{a}\cdot\mathbf{b})^2 \leq a^2b^2$, simply states that $\cos^2\theta \leq 1$.

Definition: a system of $r$ functions $f_1, \ldots, f_r$ is *linearly dependent* if

$$\sum_{i=1}^{r} c_i f_i = 0 \tag{5.8}$$

holds for all $x$ with $c_i$ not all of which are zeros.

Otherwise, the system of functions is *linearly independent*, i.e., if (5.8) is possible only when all $c_i = 0$.

Theorem: an orthogonal system of functions is always linearly independent.

Gram-Schmidt orthogonalisation.

From an infinite system of functions $v_1, v_2, \ldots,$ any $r$ of which are linearly independent for any $r$, an orthonormal system $\varphi_1, \varphi_2, \ldots,$ can be constructed by taking $\varphi_n$ as an appropriate linear combination of $v_1, \ldots, v_n$.

$$\begin{aligned}
&\varphi_1 = v_1/\|v_1\|, \\
&\varphi_2 = \tilde{v}_2/\|\tilde{v}_2\|, \quad \text{where} \quad \tilde{v}_2 = v_2 - (\varphi_1, v_2)\varphi_1, \\
&\varphi_3 = \tilde{v}_3/\|\tilde{v}_3\|, \quad \text{where} \quad \tilde{v}_3 = v_3 - (\varphi_1, v_3)\varphi_1 - (\varphi_2, v_3)\varphi_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\varphi_n = \tilde{v}_n/\|\tilde{v}_n\|, \quad \text{where} \quad \tilde{v}_n = v_n - (\varphi_1, v_n)\varphi_1 - \ldots - (\varphi_{n-1}, v_n)\varphi_{n-1}.
\end{aligned}$$

The definition of the inner product (5.1) can be generalised by including the *weight function* $\rho(x) \geq 0$ on $[a, b]$:

$$(f, g) = \int_a^b f(x)g(x)\rho(x)dx. \tag{5.9}$$

Note that all the definitions and theory above hold. In particular, if $(f, g) = 0$, we say that $f$ and $g$ are orthogonal with the weight function $\rho$.

The definition of the inner product can also be generalised for *complex* functions:

$$(f, g) = \int_a^b f^*(x)g(x)dx. \tag{5.10}$$

This quantity is in general complex, $(f, g) = (g, f)^*$, but the norm $\|f\|$,

$$\|f\|^2 = (f, f) = \int_a^b f^*(x)f(x)dx = \int_a^b |f(x)|^2 dx, \tag{5.11}$$

is obviously real and non-negative.

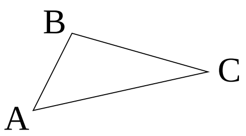
With this generalisation one can consider complex orthonormal systems of functions that satisfy (5.7). An example of such system on $[0, 2\pi]$ is

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{e^{ix}}{\sqrt{2\pi}}, \quad \frac{e^{-ix}}{\sqrt{2\pi}}, \quad \frac{e^{2ix}}{\sqrt{2\pi}}, \quad \frac{e^{-2ix}}{\sqrt{2\pi}}, \quad \ldots \ .$$

## 5.2 Expansion of a function in an orthogonal system

Let $\varphi_i(x)$, $i = 1, 2, \ldots$, be an orthonormal system. We want to approximate $f(x)$ by $\sum_{i=1}^{n} c_i\varphi_i(x)$. How to choose $c_i$ to get the best approximation?

We can use the norm $\|f - g\|$ as a measure of "distance" between two functions. It has the same properties as the distance between two points, i.e.,




$|AC| \leq |AB| + |BC|$

1. $\|f - g\| \geq 0$,
2. $\|f - g\| \leq \|f - h\| + \|h - g\|$ (the triangle inequality).[12]

To find $c_i$ that minimise $\|f - \sum_{i=1}^{n} c_i\varphi_i\|$, we show that

$$\left\|f - \sum_{i=1}^{n} c_i\varphi_i\right\|^2 = \|f\|^2 + \sum_{i=1}^{n}[c_i - (\phi_i, f)]^2 - \sum_{i=1}^{n}(\phi_i, f)^2. \tag{5.12}$$

The right-hand side takes its smallest value if

$$c_i = (\phi_i, f). \tag{5.13}$$

In this case the right-hand side of (5.12) equals $\|f\|^2 - \sum_{i=1}^{n} c_i^2$, and since the left-hand side of (5.12) is non-negative, we have

$$\sum_{i=1}^{n} c_i^2 \leq \|f\|^2. \tag{5.14}$$

This is *Bessel's inequality.* It shows that the series $\sum_{i=1}^{\infty} c_i^2$ converges.

If this series converges to $\|f\|^2$, (5.14) becomes *Parseval's relation*,

$$\sum_{i=1}^{\infty} c_i^2 = \|f\|^2, \tag{5.15}$$

which means that

$$\left\|f - \sum_{i=1}^{\infty} c_i\varphi_i\right\|^2 = 0. \tag{5.16}$$

We then say that $\sum_{i=1}^{\infty} c_i\varphi_i$ converges to $f$ *in the mean.* Written explicitly using (5.4) and (5.1), equation (5.16) reads:

$$\int_a^b \left[f(x) - \sum_{i=1}^{\infty} c_i\varphi_i(x)\right]^2 dx = 0. \tag{5.17}$$

However, this does not mean that

$$f(x) = \sum_{i=1}^{\infty} c_i\varphi_i(x) \tag{5.18}$$

[12] To show this, consider $\|f - g\|^2 = (f - g, f - g) = (f - h + h - g, f - h + h - g)$ and expand; then use Schwartz's inequality for $(f - h, h - g)$.

for all values of $x$ (e.g., recall the Gibbs phenomenon).

Definition. If $\sum_{i=1}^{\infty} \varphi_i$ converges in the mean for every piecewise continuous $f$, then we say that the system of functions $\varphi_i$, $i = 1, 2, \ldots$, is *complete*.

It serves as a *basis* in which any well-behaved function can be expanded.[13]

## 5.3 Origins of the Sturm-Liouville problem

Consider a second-order linear PDE for $u(x,t)$,

$$L[u] = \rho u_{tt}, \tag{5.19}$$

where $\rho(x) \geq 0$, and

$$L[u] = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u, \tag{5.20}$$

with $p(x) > 0$.[14] Solving (5.19) by variable separation, $u(x,t) = v(x)g(t)$, we obtain

$$g''(t) + \lambda g(t) = 0. \tag{5.21}$$

and

$$L[v] = -\lambda \rho v(x), \tag{5.22}$$

where $\lambda$ is the separation constant. Equation (5.22),

$$\frac{d}{dx}\left(p\frac{dv}{dx}\right) - qv + \lambda \rho v = 0, \tag{5.23}$$

must be solved with appropriate boundary conditions, e.g.,

$$v(a) = 0, \quad v(b) = 0, \tag{5.24}$$

for the fixed ends, or

$$v'(a) = 0, \quad v'(b) = 0, \tag{5.25}$$

for the free ends. This gives the values of $\lambda$ and the corresponding functions $v(x)$. This is the *Sturm-Liouville problem.*

[13]Note the similarity between this and the expansion of a vector $\mathbf{a}$ in an orthonormal basis $\mathbf{e}_i$ in $n$-dimensional space. Here the scalar product takes the place of the inner product, and the following familiar relations,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \qquad \mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad a_i = \mathbf{e}_i \cdot \mathbf{a},$$

and Pythagoras's theorem,

$$\mathbf{a}^2 = \sum_{i=1}^{n} a_i^2,$$

are the analogues of equations (5.7), (5.18), (5.13) and (5.15), respectively.

[14]For $q = 0$ and $p(x) = T$ this gives the wave equation for the string.

This problem also appears when solving the heat equation,[15]

$$L[u] = u_t, \tag{5.26}$$

for which variable separation leads to

$$g'(t) = -\lambda g(t), \tag{5.27}$$

and

$$L[v] + \lambda v(x) = 0. \tag{5.28}$$

The latter must be solved subject to boundary conditions, e.g.,

$$\alpha_a v(a) + \beta_a v'(a) = 0, \quad \alpha_b v(b) + \beta_b v'(b) = 0, \tag{5.29}$$

which generalise (5.24) and (5.25) (see Sec. 1.5), yielding $\lambda$ and $v(x)$.

Another example is the circular membrane problem (Sec. 2.3), which led to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)R(r) = 0. \tag{5.30}$$

Multiplying this equation by $r$, we have

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{m^2}{r}R + k^2 R = 0, \tag{5.31}$$

which has the form of (5.23), with $p = r$, $q = m^2/r$, $\rho = r$, and $\lambda = k^2$, and must be solved on $0 \le r \le a$.

Note that in this problem $r = 0$ is a singular point, since $p$ vanishes here. This makes the Sturm-Liouville problem *singular*, and we require that the solution is bounded (finite), or its growth is limited at this point. The point $r = a$ is not singular, and we used $R(a) = 0$ here, as in (5.24).

## 5.4 Self-adjoint operators. Green's formula

Consider a general 2nd-order linear differential expression,

$$L[u] = pu'' + ru' - qu, \tag{5.32}$$

where $p$, $r$ and $q$ are functions of $x$. Using integration by parts we can show that for two functions $u$ and $v$,

$$\int_a^b (vL[u] - uL[v])dx = \int_a^b (p' - r)(uv' - vu')dx + [vpu' - upv']_a^b \tag{5.33}$$

For $p' = r$ the integral contribution on the right-hand-side vanishes. Then also $L[u] = pu'' + p'u' - qu$, which can be written as

$$L[u] = (pu')' - qu. \tag{5.34}$$

[15] $L[u] = Ku_{xx}$ for the simplest case of a uniform rod.

This expression is said to be *self-adjoint*. As seen from equation (5.33), it satisfies *Green's formula*,

$$\int_a^b (vL[u] - uL[v])dx = [vpu' - upv']_a^b. \tag{5.35}$$

The typical boundary conditions imposed on $u$ (and $v$) are:

1. $u(a) = 0$, $u(b) = 0$,
2. $u'(a) = 0$, $u'(b) = 0$,
3. $\alpha_a u(a) + \beta_a u'(a) = 0$, $\alpha_b u(b) + \beta_b u'(b) = 0$,
4. $u(a) = u(b)$, $p(a)u'(a) = p(b)u'(b)$ [for $p(a) = p(b)$ these are *periodic* boundary conditions).

When $u$ and $v$ satisfy any one of these four *homogeneous*[16] boundary conditions, the right-hand side of equation (5.35) vanishes,

$$\int_a^b (vL[u] - uL[v])dx = 0. \tag{5.36}$$

This differential operator $L$ (with the boundary conditions) is *self-adjoint*.[17]

Comments:

1. If the Sturm-Liouville problem is singular, i.e., $p(a) = 0$ (or $p(b) = 0$), the right-hand side of Green's formula (5.35) will vanish without any specific boundary conditions on $u$ and $v$ at this point (as long as $u$ and $v$ are bounded or do not grow too fast).

2. One can always make a second-order linear differential expression,

$$\tilde{p}u'' + \tilde{r}u' - \tilde{q}u, \tag{5.37}$$

self-adjoint, by multiplying it by $R(x)$, and requiring

$$(R\tilde{p})' = R\tilde{r}. \tag{5.38}$$

Solving this equation gives

$$R(x) = e^{\int [(\tilde{r}-\tilde{p}')/\tilde{p}]dx}. \tag{5.39}$$

## 5.5 The Sturm-Liouville problem

The Sturm-Liouville (SL) problem requires one to solve the differential equation

$$(pu')' - qu + \lambda\rho u = 0, \tag{5.40}$$

[16]If they are satisfied by $u$, they are also satisfied by $cu$, where $c$ is a constant.

[17]Its action "on the left" and "on the right" is the same. It is analogous to a symmetric square matrix $a_{ij} = a_{ji}$, for which $\sum_{i,j=1}^{n}(v_i a_{ij} u_j - u_i a_{ij} v_j) = 0$.

on $[a, b]$ where $p(x) > 0$ (or $p(x) \geq 0$ in the singular case) and $\rho(x) \geq 0$, subject to one of the four types of boundary conditions, Sec. 5.4. This means finding *all* values of $\lambda$ for which nontrivial solutions $u$ exist. These $\lambda$ are known as the *eigenvalues* of the SL problem, and $u$ are the *eigenfunctions*.

Properties of the SL problem.

1. All eigenvalues are real (for real $p$, $q$ and $\rho$).

   Proof. Consider the SL equation,

   $$L[u] + \lambda \rho u = 0,$$

   and its complex conjugate,

   $$L[u^*] + \lambda^* \rho u^* = 0.$$

   Multiply the former by $u^*$ and the latter by $u$, subtract and integrate between $a$ and $b$, to obtain:

   $$\int_a^b (u^* L[u] - u L[u^*]) dx = (\lambda^* - \lambda) \int_a^b \rho u^* u dx. \tag{5.41}$$

   The left-hand side equals zero by (5.36), and $\int_a^b \rho |u|^2 dx > 0$. Hence, $\lambda^* = \lambda$, which means that $\lambda$ is real.

2. For the boundary conditions 1–3 the eigenvalues are *nondegenerate*, i.e., only one eigenfunction corresponds to each $\lambda$.[18]

   Proof by contradiction: let $u_1$ and $u_2$ correspond to the same $\lambda$,

   $$(pu_1')' - qu_1 + \lambda \rho u_1 = 0,$$
   $$(pu_2')' - qu_2 + \lambda \rho u_2 = 0.$$

   Multiplying these by $u_2$ and $u_1$, respectively, and subtracting, we have

   $$u_2 (pu_1')' - u_1 (pu_2')' = 0,$$

   which can be written as

   $$\frac{d}{dx}(u_2 p u_1' - u_1 p u_2') = 0,$$

   and hence

   $$u_2 p u_1' - u_1 p u_2' = \text{const}.$$

   Applying the boundary-conditions 1–3 gives const $= 0$, so that

   $$u_2 u_1' - u_1 u_2' = 0.$$

   Integrating, we obtain $u_1 = C u_2$, i.e., the two functions differ by a constant factor.

[18] To within multiplication by a constant.

3. The eigenfunctions corresponding to different eigenvalues are orthogonal on $[a, b]$ with the weight function $\rho$.

   Proof. Let $u$ and $v$ be two eigenfunctions with eigenvalues $\lambda$ and $\mu$:

$$L[u] + \lambda\rho u = 0,$$
$$L[v] + \mu\rho v = 0.$$

   Multiplying these by $v$ and $u$, respectively, subtracting and integrating, we have

$$\int_a^b (vL[u] - uL[v])dx = (\mu - \lambda)\int_a^b uv\rho dx. \tag{5.42}$$

   The left-hand side is zero by (5.36), and if $\lambda \neq \mu$ then

$$\int_a^b uv\rho dx = 0.$$

4. For $q \geq 0$ and boundary conditions 1–4 (with the additional requirement $\alpha_a/\beta_a \leq 0$, $\alpha_b/\beta_b \geq 0$ in 3), the eigenvalues are non-negative.

   Proof. Write the SL equation as

$$\lambda\rho u = -(pu')' + qu.$$

   Multiplying by $u$, integrating, and using integration by parts, we have

$$\lambda\underbrace{\int_a^b \rho u^2 dx}_{>0} = \underbrace{\int_a^b (pu'^2 + qu^2)dx}_{\geq 0} - [pu'u]_a^b.$$

   The last term on the right is zero for boundary conditions $1, 2, 4$, and nonnegative for 3, hence, $\lambda \geq 0$.

The main theorem. The eigenvalues of the SL problem, when ordered, form a countable infinite sequence, $\lambda_1 < \lambda_2 < \ldots$, and the eigenfunctions $u_n$ form a complete orthogonal system on $[a, b]$ with the weight function $\rho$.

Any function which satisfies the boundary conditions and has a continuous first and piecewise continuous second derivative on $[a, b]$, may be expanded in a uniformly and absolutely convergent series in the eigenfunctions,

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x), \tag{5.43}$$

with the coefficients given by

$$c_n = \int_a^b f(x)u_n(x)\rho(x)dx, \tag{5.44}$$

if the eigenfunctions are normalised, $\int_a^b u_m u_n \rho dx = \delta_{nm}$.

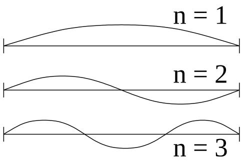
Expansion (5.43) is a (generalised) Fourier series. The above theorem ensures that we can obtain solutions of PDE in the form of such expansions, where the coefficients in the expansion depend on other variables (e.g., time):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x)[a_n \cos \omega_n t + b_n \sin \omega_n t], \tag{5.45}$$

for the wave equation ($\omega_n = \sqrt{\lambda_n}$), and

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x) c_n e^{-\lambda_n t}, \tag{5.46}$$

for the heat equation.




Oscillation theorem. The $n$th eigenfunction of the SL problem has $n-1$ zeros inside $[a,b]$. This is illustrated for the string on the left.

Fourier-Bessel expansion.

Solving the problem of vibrations for a circular membrane of radius $a$ by variable separation, we obtained $u(r,\varphi,t)$ in the form (2.24).

We now recognise that the Bessel functions $J_m(k_{m,n} r)$ with $k_{m,n} = z_{m,n}/a$ ($z_{m,n}$ being the $n$th root of the $J_m$) are the eigenfunctions of the Sturm-Liouville problem (5.31) with the boundary conditions $R(0)$ finite, $R(a) = 0$, and weight function $r$. The main theorem tells us that $J_m(z_{m,n} r/a)$, $n = 1, 2, \ldots$, form a complete orthogonal system on $[0,a]$, so that

$$\int_0^a J_m\left(\frac{z_{m,n} r}{a}\right) J_m\left(\frac{z_{m,n'} r}{a}\right) r dr = 0 \quad \text{for} \quad n \neq n'. \tag{5.47}$$

Any well-behaved function can be expanded in $J_m$ as

$$f(r) = \sum_{n=1}^{\infty} C_n J_m\left(\frac{z_{m,n} r}{a}\right), \tag{5.48}$$

with the coefficients gives by

$$C_n = \frac{\displaystyle\int_0^a f(r) J_m\left(\frac{z_{m,n} r}{a}\right) r dr}{\displaystyle\int_0^a \left[J_m\left(\frac{z_{m,n} r}{a}\right)\right]^2 r dr}. \tag{5.49}$$

This is known as the *Fourier-Bessel* expansion.

To evaluate the squared norm $\left\| J_m\left(\frac{z_{m,n} r}{a}\right)\right\|^2$ in the denominator of (5.49), multiply (5.31) by $rR'$ and integrate between 0 and $a$,

$$\int_0^a rR'(rR')' dr - m^2 \int_0^a RR' dr + k^2 \int_0^a rRrR' dr = 0. \tag{5.50}$$

Using $ff'dr = d\left(\frac{1}{2}f^2\right)$ in all three terms, integrating by parts in the third, and taking into account the boundary conditions, we obtain

$$\int_0^a R^2 r dr = \frac{a^2}{2k^2}[R'(a)]^2. \tag{5.51}$$

For $k = k_{m,n}$ and $R(r) = J_m(k_{m,n}r)$, this gives

$$\left\| J_m\left(\frac{z_{m,n}r}{a}\right)\right\|^2 \equiv \int_0^a \left[J_m(k_{m,n}r)\right]^2 r dr = \frac{a^2}{2}\left[J'_m(k_{m,n}r)\right]^2. \tag{5.52}$$

<u>Note:</u> using (2.21), it is easy to show that

$$\frac{d}{dz}\left[\frac{J_m(z)}{z^m}\right] = -\frac{J_{m+1}(z)}{z^m}, \qquad \frac{d}{dz}[z^m J_m(z)] = z^m J_{m-1}(z),$$

and hence, express $J'_m(z)$ in terms of the Bessel functions themselves, e.g.,

$$J'_m(z) = \frac{1}{2}[J_{m-1}(z) - J_{m+1}(z)].$$

Let us use the Fourier-Bessel expansion to determine the motion of a membrane of radius $a$, struck at a point on the $\varphi = 0$ line, a distance $b < a$ from the centre.

We need to solve the wave equation in plane polar coordinates (2.11) for $u(r, \varphi, t)$ subject to the boundary condition $u(a, \varphi, t) = 0$ (as in Sec. 2.3), with the initial conditions,

$$u(r, \varphi, 0) = 0, \quad u_t(r, \varphi, 0) = \frac{P}{\rho b}\delta(r - b)\delta(\varphi), \tag{5.53}$$

where $\rho$ is the mass density of the membrane, and $P$ is the momentum imparted to the membrane at $r = b$ and $\phi = 0$, at $t = 0$.[19]

We first write the solution as a linear superposition of (2.24),

$$u(r, \varphi, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{m,n}r)(A_{m,n}\cos m\varphi + B_{m,n}\sin m\varphi)\cos(\omega_{m,n}t + \alpha_{m,n}), \tag{5.54}$$

with arbitrary $A_{m,n}$, $B_{m,n}$ and $\alpha_{m,n}$. We then find these coefficients from the initial conditions: $\alpha_{m,n} = -\pi/2$, $B_{m,n} = 0$, and

$$A_{m,n} = \frac{\frac{P}{\rho}J_m(k_{m,n}b)}{\pi(1 + \delta_{m0})\omega_{n,m}\left\|J_m\left(\frac{z_{m,n}r}{a}\right)\right\|^2}. \tag{5.55}$$

Hence, the motion of the membrane is described by

$$u(r, \varphi, t) = \frac{2P}{\pi\rho a c}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{J_m\left(\frac{z_{m,n}b}{a}\right)J_m\left(\frac{z_{m,n}r}{a}\right)}{z_{m,n}(1 + \delta_{m0})\left[J'_m(z_{m,n})\right]^2}\cos m\varphi \sin\frac{z_{m,n}ct}{a}. \tag{5.56}$$

[19] The second condition can be verified by calculating the momentum and using the properties of the $\delta$-function, $\int_{-\pi}^{\pi}\int_0^a \rho u_t(t, \varphi, 0) r dr d\varphi = P$.

# 6 Normal forms of the 2nd-order PDE in two variables

## 6.1 Hyperbolic, parabolic and elliptic types of equations

Recall that when solving similar problems, e.g., the wave equation,

$$c^2 u_{xx} - u_{tt} = 0 \text{ for } u(x,t) \text{ on } -\infty < x < \infty,\ 0 \le t < \infty,$$

and Laplace's equation,

$$u_{xx} + u_{yy} = 0 \text{ for } u(x,y) \text{ on } -\infty < x < \infty,\ 0 \le y < \infty,$$

we used different types or boundary/initial conditions. Namely, in the first case, we used $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ (Sec. 4.2, Problem 2), while in the second case (Problem sheet 5, Example 1), it was sufficient to set $u(x,0) = f(x)$. This difference comes from the fact that, though the equations only differ by a sign, minus or plus, their solutions behave very differently. In the case of Laplace's equation, an additional implicit condition was that the solution vanishes at infinity.

In this chapter we learn how to distinguish such *types* of 2nd-order equations.

A second-order linear or quasi-linear[20] equation in two variables, $x$ and $y$,

$$a u_{xx} + 2b u_{xy} + c u_{yy} + g(x,y,u,u_x,u_y) = 0, \tag{6.1}$$

where $a$, $b$ and $c$ are functions or $x$ and $y$, can be transformed to new independent variables,

$$\xi = \phi(x,y), \quad \eta = \psi(x,y), \tag{6.2}$$

using the following expressions for the derivatives,

$$\begin{aligned}
u_x &= u_\xi \phi_x + u_\eta \psi_x, \qquad u_y = u_\xi \phi_y + u_\eta \psi_y, & (6.3)\\
u_{xx} &= u_{\xi\xi}\phi_x^2 + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x^2 + u_\xi\phi_{xx} + u_\eta\psi_{xx} & (6.4)\\
u_{xy} &= u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + u_\xi\phi_{xy} + u_\eta\psi_{xy} & (6.5)\\
u_{yy} &= u_{\xi\xi}\phi_y^2 + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y^2 + u_\xi\phi_{yy} + u_\eta\psi_{yy}. & (6.6)
\end{aligned}$$

Equation (6.1) then assumes the form

$$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \tilde{g}(\xi,\eta,u,u_\xi,u_\eta) = 0, \tag{6.7}$$

where the new coefficients are

$$\begin{aligned}
\alpha &= a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2, & (6.8)\\
\beta &= a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y, & (6.9)\\
\gamma &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. & (6.10)
\end{aligned}$$

[20] *Quasi-linear* means that the equation is linear with respect to the 2nd-order derivatives.

It is easy to verify that

$$\beta^2 - \alpha\gamma = (b^2 - ac)(\phi_x \psi_y - \phi_y \psi_x)^2, \tag{6.11}$$

which shows that the *sign* of $\beta^2 - \alpha\gamma$ is the same as that of $b^2 - ac$ for any transformation.[21]

Depending on $\Delta = b^2 - ac$, we classify the equation as

1. $\Delta > 0$, *hyperbolic*,
2. $\Delta = 0$, *parabolic*,
3. $\Delta < 0$, *elliptic*.

Examples: the wave equation $u_{tt} - c^2 u_{xx} = 0$ is hyperbolic, the heat equation, $u_t - K u_{xx} = 0$, is parabolic, and Laplace's equation, $u_{xx} + u_{yy} = 0$, is elliptic.

## 6.2 Reduction to the normal form

Consider an auxiliary quadratic equation,

$$a\lambda^2 + 2b\lambda + c = 0. \tag{6.12}$$

Note that it is satisfied by $\lambda = \phi_x / \phi_y$ if we set $\alpha = 0$ [equation (6.8)], and by $\lambda = \psi_x / \psi_y$ if we set $\gamma = 0$ [equation (6.10)]. The roots of (6.12) are

$$\lambda_{1,2} = \frac{1}{a}(-b \pm \sqrt{\Delta}). \tag{6.13}$$

Depending on its type, we reduce the PDE to the *normal* (or *canonical*) form as follows:

1. Hyperbolic, $\Delta > 0$.

   Equation (6.12) has two distinct real roots. For the normal form we require $\alpha = \gamma = 0$; $\xi$ and $\eta$ are found by solving

   $$\phi_x - \lambda_1 \phi_y = 0, \qquad \psi_x - \lambda_2 \psi_y = 0. \tag{6.14}$$

   and the PDE is written in the normal form as $u_{\xi\eta} + \ldots = 0$.[22]

2. Parabolic, $\Delta = 0$.

   For the normal form we require $\alpha = \beta = 0$; the variable $\xi$ is found from

   $$a\phi_x + b\phi_y = 0,$$

   and $\eta$ is arbitrary, independent of $\xi$, such that $\gamma \neq 0$ (e.g., it may be possible to set $\eta = x$ or $\eta = y$). The normal form reads $u_{\eta\eta} + \ldots = 0$.

[21] Note that $\phi_x \psi_y - \phi_y \psi_x$ that appears on the right-hand side of (6.11) is the *Jacobian* of the variable transformation (6.2), which must be nonzero.

[22] To solve equation of the form $\frac{\partial \phi}{\partial x} + p(x,y)\frac{\partial \phi}{\partial y} = 0$, solve the ordinary differential equation $\frac{dy}{dx} = p(x,y)$, and write its solution as $\phi(x,y) = \text{const}$.

3. Elliptic, $\Delta < 0$.

   Equation (6.12) has complex conjugate roots. Solving the first of equations (6.14) we find a complex $\xi$, and set $\eta = \xi^*$. Regarding these variables as independent, we obtain the PDE in the complex normal form $u_{\xi\eta} + \ldots = 0$.

   Introducing real variables, $\rho = \dfrac{\xi + \eta}{2}$ and $\sigma = \dfrac{\xi - \eta}{2i}$, we obtain $u_{\xi\eta} = \frac{1}{4}(u_{\rho\rho} + u_{\sigma\sigma})$, and write the elliptic PDE in the real normal form as $u_{\rho\rho} + u_{\sigma\sigma} + \ldots = 0$.