# Partial Differential Equations AMA3006 — Problem sheet 1

## ODE and simple PDE. Initial and boundary conditions.

### Examples

1. Solve the differential equation
$$\frac{d^2y}{dx^2} + \lambda y = 0,$$
where $\lambda$ is a constant, subject to the boundary conditions $y'(0) = y(l) = 0$.

   Find all values of $\lambda$ for which this problem has nontrivial (i.e., nonzero) solutions.

2. Find a partial differential equation satisfied by all functions of the form $u = f(x^2 - y^2)$. (Assume $f$ to have as many derivatives as needed.)

3. Find the general solution $u(x, y, z)$ of $u_{xxx} + u_x = 0$.

4. At the point $x = c$ of a string ($0 \le x \le l$) a concentrated weight of mass $m_0$ is fixed. Write down the equations defining the vibration process for arbitrary initial conditions, assuming that the ends of the string are fixed.

### Homework problems

1. Solve the following ordinary differential equations (ODE):

   (a) $\dfrac{dx}{dt} = -\lambda x$, where $\lambda$ is a constant, and $x(0) = x_0$,

   (b) $y' + y \tan x = \cos x$,

   (c) $y'' + 4y' + 5y = \sin 2x$.

   In (a) find the solutions that satisfies the boundary condition given. In (b) and (c) identify the complementary functions and particular integrals in the general solutions.

2. Solve the differential equation
$$\frac{d^2y}{dx^2} + \lambda y = 0,$$
where $\lambda$ is a constant, subject to the boundary conditions $y(0) = y(1) = 0$.

   Find all values of $\lambda$ for which this problem has nontrivial (i.e., nonzero) solutions.

3. For each of the following partial differential equations (PDE), state its order, whether the equation is linear or nonlinear, and if it is linear, whether it is homogeneous or inhomogeneous:

   (a) $u_x u_y - 3u = x$

   (b) $y u_{xxy} - e^x u_x + 3 = 0$

   (c) $\rho u_{tt} + \mu u_{xxxx} = 0$

   (d) $u u_{xy} - u_x u_y = 0$

   (e) $u_{xx} + u_{yy} + \log u = \log x$.

4. Show that every function $u$ given by $u = f(x^2 + y^2)$, where $f$ is an arbitrary function of one variable having a continuous derivative, is a solution of $y u_x - x u_y = 0$.

5. Find a partial differential equation satisfied by all functions of the form

(a) $u = f(xy)$

(b) $u = f(x^2 + y^2) + g(x)$.

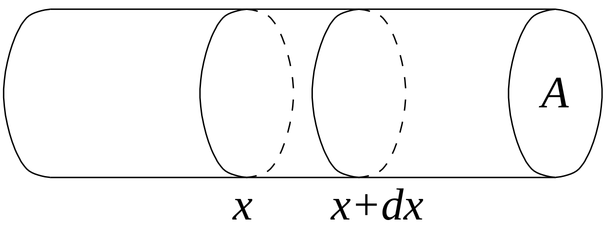
(Assume $f$ and $g$ to have as many derivatives as needed for your argument.)

6. Find the general solution $u(x, y)$ of the differential equations

   (a) $u_{xxy} = 1$

   (b) $u_{xx} - u = 0$.

7. Derive the heat equation for the uniform rod of cross section $A$, density $\rho$ and specific heat $c$, whose temperature $T$ depends only on the coordinate $x$ along the rod.




   Hints: for a slice of the rod of thickness $dx$, consider the heat energy that flows into this slice at $x$ and out of it at $x + dx$, in time $dt$, and relate their difference to a change in the amount of heat energy between $x$ and $x + dx$. According to the Fourier law, for the heat conduction in one dimension, the heat energy flux density is given by

$$j = -\kappa \frac{\partial T}{\partial x},$$

   where $\kappa$ is the thermal conductivity.

8. Write down the initial and boundary conditions for the problem on the vibrations of a string with fixed ends ($0 \leq x \leq l$), which at the initial moment of time $t = 0$ is pulled at the point $x = c$ to a given value $h$ and released without an initial velocity.