Partial Differential Equations AMA3006 — Problem sheet 2

Variable separation method.

Examples

1. Using variable separation, solve Laplace's equation $\nabla^2 u = 0$ in two dimensions using plane polar coordinates,
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0,$$
and show that a solution of this equation can be constructed as
$$u(r,\varphi) = C\ln r + D + \sum_{n=1}^{\infty}(A_n\cos n\varphi + B_n\sin n\varphi)(E_n r^n + F_n r^{-n}),$$
where C, D, A_n, B_n, E_n and F_n are arbitrary constants.

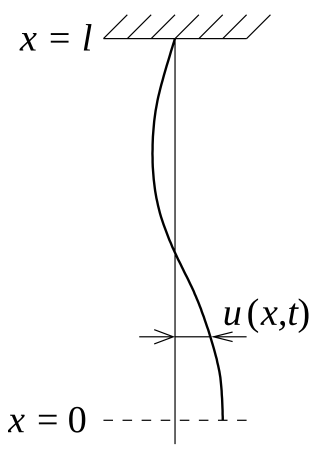
2. Prove that
$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = \begin{cases} 0 & n=m=0 \\ \frac{1}{2}l\delta_{nm} & \text{otherwise} \end{cases} \tag{1}$$
$$\int_{-l}^l \sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0 \qquad \text{(note the limits)} \tag{2}$$
$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = \begin{cases} l & n=m=0 \\ \frac{1}{2}l\delta_{nm} & \text{otherwise,} \end{cases} \tag{3}$$
for integer $n, m \geq 0$, where $\delta_{nm} = 1$ for $n = m$, 0 for $n \neq m$, is the *Kronecker* delta symbol.

Homework problems

1. Using the form $u(x,t) = v(x)q(t)$, solve the one-dimensional wave equation,
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$
for the string of length l ($0 \leq x \leq l$) with boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ (i.e., fixed end at $x = 0$ and "free" end at $x = l$).

   Hence, show that the string can execute harmonic vibrations described by
$$u(x,t) = A\sin\left[\pi(n+\tfrac{1}{2})x/l\right]\cos(\omega_n t + \phi),$$
with frequencies $\omega_n = \pi c(n+\frac{1}{2})/l$, $n = 0, 1, \ldots,$ and arbitrary amplitude A and phase ϕ.

2. A rope of length l and linear mass density ρ hangs freely along the x axis under gravity (acceleration g). The bottom end of the string lies at $x = 0$ and the top at $x = l$.



   (a) Using the approach used for the string, show that the displacement $u(x,t)$ of the rope satisfies the equation
$$\frac{\partial^2 u}{\partial t^2} - g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = 0. \tag{4}$$
[Hint: at point x the tension force in the rope is $T = g\rho x$.]

   (b) Seeking solution of Eq. (4) in the form $u(x,t) = v(x)q(t)$, find $q(t)$ and show that $v(x)$ satisfies the equation
$$x\frac{d^2 v}{dx} + \frac{dv}{dx} + \frac{\omega^2}{g}v = 0, \tag{5}$$
where $-\omega^2$ is the separation constant.

(c) Introduce a new independent variable $\xi = \alpha\sqrt{x}$, i.e., $x = \xi^2/\alpha^2$, where α is a constant, and show that Eq. (5) takes the form

$$\frac{d^2v}{d\xi^2} + \frac{1}{\xi}\frac{dv}{d\xi} + v = 0, \tag{6}$$

if one chooses $\alpha = 2\omega/\sqrt{g}$.[1]

(d) Equation (6) is the Bessel equation for $m = 0$, whose regular solution is $J_0(\xi)$. Hence, show that the solutions $v(x)$ of Eq. (5) such that $v(0)$ is finite and $v(l) = 0$, are

$$v(x) = AJ_0\left(z_{0,n}\sqrt{\frac{x}{l}}\right), \quad n = 1, 2, \ldots, \tag{7}$$

where A is an arbitrary constant, $z_{0,n}$ is the nth root of $J_0(z)$, and $\omega \equiv \omega_n = \frac{z_{0,n}}{2}\sqrt{\frac{g}{l}}$.

(e) Combining the results from (a)–(d), show that the hanging rope executing harmonic motion with frequency ω_n, is described by

$$u(x,t) = AJ_0\left(z_{0,n}\sqrt{\frac{x}{l}}\right)\cos(\omega_n t + \phi),$$

where ϕ is an arbitrary initial phase.

3. Consider the one-dimensional heat equation for $0 \le x \le l$ (rod of length l),

$$u_t - Ku_{xx} = 0. \tag{8}$$

(a) Show that when the rod is in thermal equilibrium (i.e., the temperature does not change with time, $\partial u/\partial t = 0$), the time-independent (or *stationary*) solution of Eq. (8), $u_s(x)$, which satisfies the boundary conditions $u_s(0) = T_1$, $u_s(l) = T_2$, is

$$u_s(x) = T_1 + (T_2 - T_1)x/l. \tag{9}$$

(b) Show that if $u_0(x,t)$ is a solution of Eq. (8) with $u_0(0,t) = u_0(l,t) = 0$, then $u(x,t) = u_0(x,t) + u_s(x)$ satisfies Eq. (8) with boundary conditions $u(0,t) = T_1$, $u(l,t) = T_2$.

(c) Using

$$u_0(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-(n^2\pi^2/l^2)Kt}, \tag{10}$$

show that the solution which satisfies $u(0,t) = T_1$, $u(l,t) = T_2$ and the *initial* condition $u(x,0) = f(x)$, is

$$u(x,t) = T_1 + (T_2 - T_1)x/l + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-(n^2\pi^2/l^2)Kt}, \tag{11}$$

where $B_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}[f(x) - T_1 - (T_2 - T_1)x/l]dx$.

4. Using the method of separation of variables, solve the two-dimensional wave equation in Cartesian coordinates for a rectangular membrane ($0 \le x \le a$, $0 \le y \le b$) with fixed edges, $u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t)$, and show that the membrane executes harmonic motion with frequencies $\omega_{nm} = \pi c(n^2/a^2 + m^2/b^2)^{1/2}$, where $n, m = 0, 1, 2, \ldots$, and described by $u(x,y,t) = A\sin(n\pi x/a)\sin(m\pi y/b)\cos(\omega_{nm}t + \phi)$.

[1]Hint: use chain rule to transform the derivatives, $\frac{dv}{dx} = \frac{dv}{d\xi}\frac{d\xi}{dx}$, $\frac{d^2v}{dx^2} = \frac{d^2v}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 + \frac{dv}{d\xi}\frac{d^2\xi}{dx^2}$.