# AMA3006 Problem sheet 2



## SOLUTIONS

(1) One-dimensional wave equation for the string, $0 \le x \le l$:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

with boundary conditions $u(0,t) = 0$
$u_x(l,t) = 0$

[ fixed end at $x = 0$; "free" end at $x = l$ ].

Seek solution in the form:

$$u(x,t) = v(x)\, q(t) . \qquad (2)$$

Substituting into (1):

$$v(x) \frac{\partial^2 q}{\partial t^2} - q(t)\, c^2 \frac{\partial^2 v}{\partial x^2} = 0 ,$$

dividing by $v(x) q(t)$ and replacing partial derivatives by total ($\partial \to d$):

$$\underbrace{\frac{1}{q(t)} \frac{d^2 q}{dt^2}}_{\text{function of } t \text{ only}} - \underbrace{\frac{c^2}{v(x)} \frac{d^2 v}{dx^2}}_{\text{function of } x \text{ only}} = 0 . \qquad (3)$$

The two terms in (3) must be constants, i.e., denoting the constant $-\omega^2$:

$$\frac{1}{q(t)} \frac{d^2 q}{dt^2} = -\omega^2 . \qquad (4)$$

[We will see below that the separation constant must be negative.] Eq. (4) gives: $q'' + \omega^2 q = 0$, So, the general solution of (4) is

$$q(t) = C_1 \cos \omega t + C_2 \sin \omega t,$$

which can also be written as

$$q(t) = A \cos(\omega t + \phi),$$ where $\phi$ is the initial phase.

Turning to the coordinate part of Eq. (3) now:

$$\frac{c^2}{v(x)} \frac{d^2 v}{dx^2} = -\omega^2$$

$$\frac{d^2 v}{dx^2} = -\frac{\omega^2}{c^2} v(x) \;;$$

Denoting $\frac{\omega}{c} \equiv k$, we have:

$$v'' + k^2 v = 0 \; . \qquad (5)$$

The general solution of this equation is:

$$v(x) = C_1 \cos kx + C_2 \sin kx \; .$$

Using the 1st boundary condition, $u(0,t)=0$, we must have $v(0) = 0$, hence: $0 = C_1 \cdot 1 + C_2 \cdot 0$,

and $C_1 = 0$.

Therefore $v(x) = C \sin kx$, where we have repamed $C_2 \longrightarrow C$.

Using the second boundary condition: $v'(\ell) = 0$ :

$$u_x(\ell, t) = 0$$

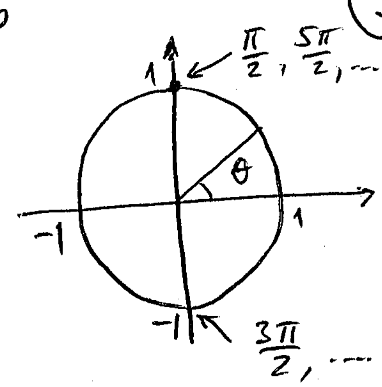
$$v'(x) = Ck \cos kx$$

$$Ck \cos k\ell = 0$$

To have a nonzero solution, we require

$$\underline{\cos k\ell = 0} \; .$$

For positive $\theta$, $\cos\theta$ vanishes
at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots,$




i.e. $\theta = \frac{\pi}{2} + n\pi,$
$n = 0, 1, 2, \ldots$

Hence $\underline{kl = \pi(n + \tfrac{1}{2})}$, $n = 0, 1, 2, \ldots$

$$k = \frac{\pi}{l}(n + \tfrac{1}{2})$$

$$v_n(x) = C \sin \frac{(n+\frac{1}{2})\pi x}{l} \tag{6}$$

The corresponding frequencies are

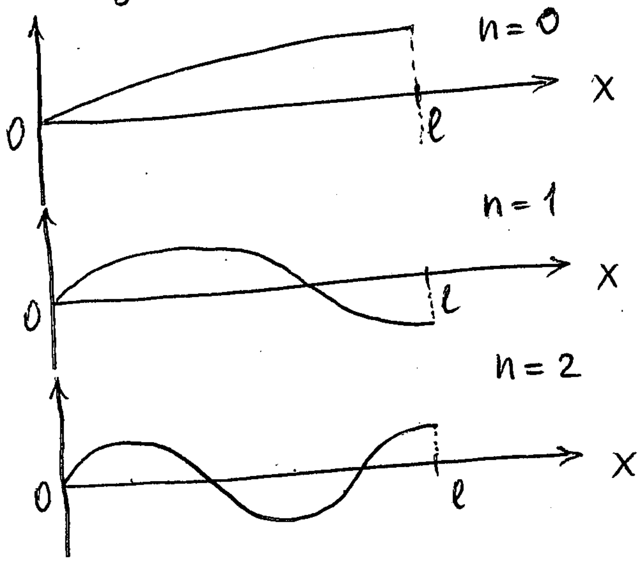
$$\omega = kc = \frac{\pi c(n+\frac{1}{2})}{l} \equiv \omega_n .$$

Combining $v(x)$, Eq. (6), with $q(t)$, we have:

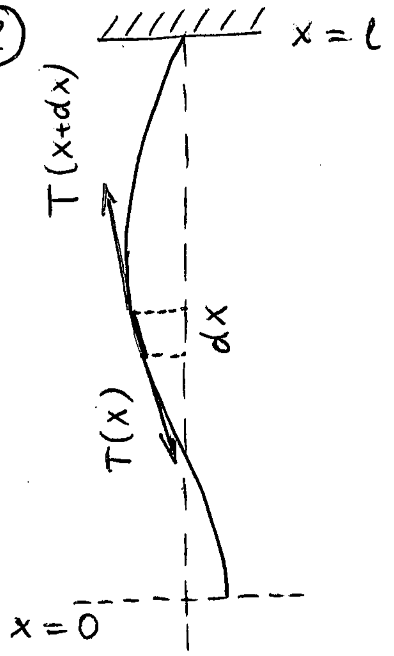
$$\boxed{u(x,t) = A \sin \frac{(n+\frac{1}{2})\pi x}{l} \cos(\omega_n t + \phi)}$$

where $A$ is an arbitrary constant.

These solutions describe harmonic vibrations with frequency $\omega_n$. They can be visualised by plotting $v_n(x)$.




In these solutions the phase of the sin function increases from 0 at $x = 0$ to $(n+\frac{1}{2})\pi$ at $x = l$. The number of nodes between $x = 0$ and $x = l$ is $n$.

② 

(a) Considering the segment of the rope of length $dx$:

its mass is $\rho\,dx$,

Forces acting on it at the two ends: $T(x+dx)$ and $T(x)$.

Taking their components in the direction perpendicular to $x$, we obtain the total force acting on $dx$ segment in this direction:

$$T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)$$

$$= \frac{\partial}{\partial x}\Big(T(x)\sin\theta(x)\Big)\,dx, \qquad (1)$$

where $\theta(x)$ is the angle the rope makes with the $x$ direction.

If the perpendicular displacement of the rope is $u(x,t)$, then, for small displacements and "nearly straight" rope, $\theta \ll 1$,

$$\sin\theta \simeq \tan\theta = \frac{\partial u}{\partial x} \qquad (2)$$

The acceleration of the segment is $\frac{\partial^2 u}{\partial t^2}$, hence by Newton's 2nd law:

$$\rho\,\cancel{dx}\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right)\cancel{dx} \qquad (3)$$

The tension in the rope at point $x$ is equal to the weight of the rope below, i.e. $T = \rho x g$.

Substituting this into (3) we obtain:

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\rho x g \frac{\partial u}{\partial x}\right)$$

Cancelling $\rho$ and shifting the term from RHS to LHS, we have:

$$\underline{\frac{\partial^2 u}{\partial t^2} - g \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) = 0,} \qquad (4)$$

as required.

(b) Seek solution of (4) in the form:

$$u(x,t) = v(x) q(t). \qquad (5)$$

Substituting (5) into (4):

$$v(x) \frac{\partial^2 q}{\partial t^2} - q(t) g \frac{\partial}{\partial x}\left(x \frac{\partial v}{\partial x}\right) = 0$$

Dividing through by $v(x)q(t)$:

$$\underbrace{\frac{1}{q(t)} \frac{d^2 q}{dt^2}}_{\text{function of } t} - \underbrace{\frac{g}{v(x)} \frac{d}{dx}\left(x \frac{dv}{dx}\right)}_{\text{function of } x} = 0$$

The two terms must be <u>constants</u>. Choosing the separation constant as $-\omega^2$, we have for the temporal part:

$$\frac{1}{q(t)} q'' = -\omega^2$$

$$q'' + \omega^2 q = 0$$

General solution:

$$q(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$$\text{or } q(t) = A \cos(\omega t + \phi),$$

where $\phi$ is the initial phase.

Turning to the spatial, coordinate part now:

$$\frac{g}{v(x)}\frac{d}{dx}\left(x\frac{dv}{dx}\right) = -\omega^2$$

$$g\frac{d}{dx}\left(x\frac{dv}{dx}\right) + \omega^2 v(x) = 0$$ Using product rule

$$g\frac{dv}{dx} + gx\frac{d^2v}{dx^2} + \omega^2 v(x) = 0 \quad :$$ divide by $g$

$$\underline{x\frac{d^2v}{dx^2} + \frac{dv}{dx} + \frac{\omega^2}{g}v = 0,} \qquad (6)$$

as required.

(c) New variable $\xi = \alpha\sqrt{x}$, i.e. $x = \xi^2/\alpha^2$.

We need to express the derivatives in Eq. (6) in terms of derivatives with respect to $\xi$.

$$\frac{dv}{dx} = \frac{dv}{d\xi}\frac{d\xi}{dx} = \frac{dv}{d\xi}\cdot\frac{\alpha}{2\sqrt{x}}$$

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(\frac{dv}{d\xi}\cdot\frac{\alpha}{2\sqrt{x}}\right)$$

Using chain rule and then product and chain rule for the 2nd derivative

$$= \frac{d}{dx}\left(\frac{dv}{d\xi}\right)\cdot\frac{\alpha}{2\sqrt{x}} + \frac{dv}{d\xi}\frac{d}{dx}\left(\frac{\alpha}{2\sqrt{x}}\right)$$

$$= \frac{d^2v}{d\xi^2}\cdot\frac{d\xi}{dx}\cdot\frac{\alpha}{2\sqrt{x}} + \frac{dv}{d\xi}\cdot\left(-\frac{\alpha}{4x^{3/2}}\right)$$

$$= \frac{d^2v}{d\xi^2}\cdot\frac{\alpha^2}{4x} - \frac{\alpha}{4x^{3/2}}\frac{dv}{d\xi}$$

Substituting the derivatives into (6) and expressing $\sqrt{x}$, $x$, $x^{3/2}$ in terms of $\xi$, we have:

$$\frac{\xi^2}{\alpha^2}\left(\frac{d^2v}{d\xi^2}\frac{\alpha^2}{4\xi^2/\alpha^2} - \frac{\alpha\,\alpha^3}{4\xi^3}\frac{dv}{d\xi}\right) + \frac{dv}{d\xi}\cdot\frac{\alpha\alpha}{2\xi} + \frac{\omega^2}{g}v = 0$$

$$\frac{\alpha^2}{4}\frac{d^2v}{d\xi^2} - \frac{\alpha^2}{4\xi}\frac{dv}{d\xi} + \frac{\alpha^2}{2\xi}\frac{dv}{d\xi} + \frac{\omega^2}{g}v = 0 \quad \Big\} \times \frac{4}{\alpha^2}$$

$$\frac{d^2v}{d\xi^2} + \frac{1}{\xi}\frac{dv}{d\xi} + \frac{4\omega^2}{\alpha^2 g}v = 0 .$$

Choosing $\frac{4\omega^2}{\alpha^2 g} = 1 \iff \underline{\alpha = \frac{2\omega}{\sqrt{g}}}$,

we have:

$$\underline{\frac{d^2v}{d\xi^2} + \frac{1}{\xi}\frac{dv}{d\xi} + v = 0}, \qquad (7)$$

as required.

(d) Equation (7) is the Bessel equation for $m = 0$.
Its solution finite at $x = 0$ ($\xi = 0$) (bottom end of the rop) is:

$$v = J_0(\xi) = J_0(\alpha\sqrt{x})$$

$$\Rightarrow \quad \underline{v = A\, J_0\left(\frac{2\omega}{\sqrt{g}}\sqrt{x}\right)} .$$

We can multiply the solution of a homogeneous equation by an arbitrary const.

At the top end, $x = l$, the string is fixed, i.e. $u(l,t) = 0$, hence, $v(l) = 0$:

$$J_0\left(\frac{2\omega}{\sqrt{g}}\sqrt{l}\right) = 0 .$$

Therefore, $\frac{2\omega}{\sqrt{g}}\sqrt{l} = z_{0,n}$, where $z_{0,n}$ is the nth root of $J_0(z)$. So, the frequency is given by

$$\omega = \frac{z_{0,n}\sqrt{g}}{2\sqrt{l}} \equiv \omega_n ,$$

or

$$\underline{\omega_n = \frac{z_{0,n}}{2}\sqrt{\frac{g}{l}}} .$$

Substituting this $\omega$ value into $v(x)$ we have:

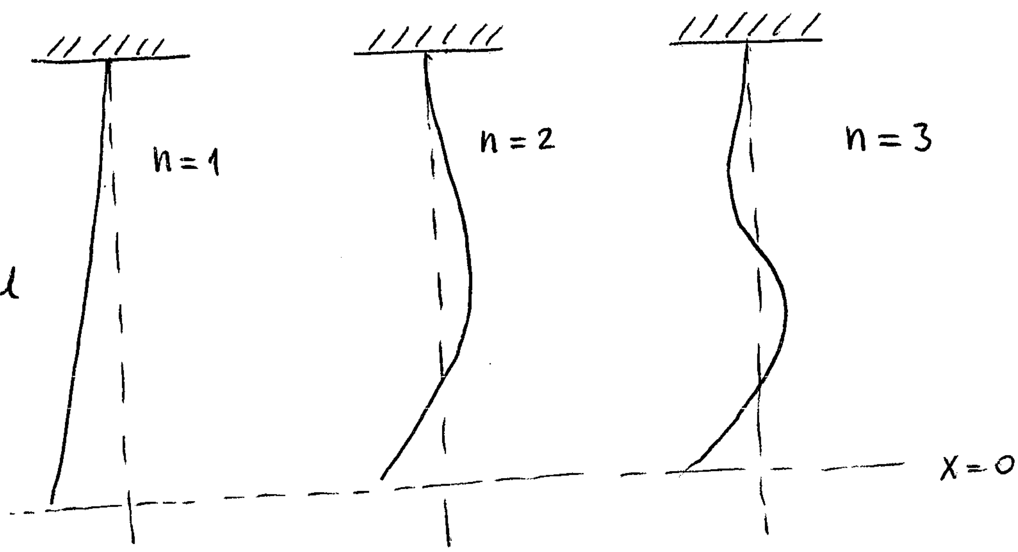
$$v(x) = A\, J_0\left(z_{0,n}\sqrt{\frac{x}{\ell}}\right).$$

(e) Combining the above expression with $q(t)$ obtained in part (b), we find:

$$u(x,t) = A\, J_0\left(z_{0,n}\sqrt{\frac{x}{\ell}}\right)\cos(\omega_n t + \phi).$$

This solution describes harmonic vibrations of the hanging rope. They can be visualised by sketching the shape of the rope for a fixed $t$.

Higher $n$ obviously correspond to higher vibrational frequencies.




③ (a)
$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

For a time independent solution $u(x,t) = u_s(x)$
$\frac{\partial u_s}{\partial t} = 0$, hence $u_s(x)$ satisfies:

$$-K\frac{d^2 u_s}{dx^2} = 0 \qquad (2)$$

[Note: $u_s$ depends on $x$ only, hence we use the total derivative, not partial.]

The solution of (2), or equivalently, $u_s'' = 0$ is a linear function:

$$u_s(x) = C_1 x + C_2 .$$

Using the boundary conditions: $u_s(0) = T_1$, $u_s(l) = T_2$:

$$C_1 \cdot 0 + C_2 = T_1 \quad \Rightarrow \quad C_2 = T_1$$

$$C_1 l + C_2 = T_2 \quad \Rightarrow \quad C_1 = (T_2 - T_1)/l$$

$$\Rightarrow \quad \underline{u_s(x) = T_1 + \frac{T_2 - T_1}{l} x}, \quad \text{as required.}$$

(b) Suppose, $u_0(x,t)$ is a solution of (1) with boundary conditions $u_0(0,t) = u_0(l,t) = 0$.

Consider: $u(x,t) = u_0(x,t) + u_s(x)$.

Substituting into (1) (left-hand side):

$$\frac{\partial}{\partial t}\left(u_0(x,t) + u_s(x)\right) - K \frac{\partial^2}{\partial x^2}\left(u_0(x,t) + u_s(x)\right)$$

$$= \underbrace{\frac{\partial u_0}{\partial t} - K \frac{\partial^2 u_0}{\partial x^2}}_{=0} - \underbrace{K \frac{\partial^2 u_s}{\partial x^2}}_{=0} = 0.$$

Hence $u = u_0 + u_s$ is a solution of the heat eq'n.

At the boundaries:
$$u(0,t) = \underbrace{u_0(0,t)}_{=0} + \underbrace{u_s(0)}_{=T_1} = T_1,$$
$$u(l,t) = \underbrace{u_0(l,t)}_{=0} + \underbrace{u_s(l)}_{=T_2} = T_2.$$

(c) We know from Sec. 2.4 that $u_0(x,t)$ can be constructed as a sum:

$$u_0(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-(n^2\pi^2/l^2)Kt}.$$

Hence, the solution that satisfies the boundary conditions $u(0,t) = T_1$, $u(l,t) = T_2$, as we have seen in part (b), can be obtained as the sum:

$$u(x,t) = \underbrace{T_1 + \frac{T_2 - T_1}{l} x}_{u_s(x)} + \underbrace{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-(n^2\pi^2/l^2)Kt}}_{u_0(x,t)}$$

To satisfy the initial condition $u(x,0)=f(x)$, we require:

$$T_1 + \frac{T_2-T_1}{\ell}x + \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{\ell} = f(x)$$

Note: $e^{-(n^2\pi^2/\ell^2)Kt}$ equals 1 for $t=0$.

$$\sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{\ell} = f(x) - T_1 - \frac{T_2-T_1}{\ell}x$$

Multiplying by $\sin\frac{m\pi x}{\ell}$ and integrating over $x$ between 0 and, we use the first of the integrals in Examples (Q.2) and see that only one term, with $n=m$, gives a nonzero contribution.

Hence:

$$B_m \frac{\ell}{2} = \int_0^{\ell} \left( f(x) - T_1 - \frac{T_2-T_1}{\ell}x \right) \sin\frac{m\pi x}{\ell}\, dx$$

$$\Rightarrow \quad B_m = \frac{2}{\ell}\int_0^{\ell} \left[ f(x) - T_1 - \frac{T_2-T_1}{\ell}x \right] \sin\frac{m\pi x}{\ell}\, dx,$$

as required.

[ $B_n$ is given by the same expression with $\sin\frac{n\pi x}{\ell}$ under the integral on the right-hand side.]

(4) Two dimensional wave equation in Cartesian coordinates $x$ and $y$ is

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \qquad (1)$$

for $u(x,y,t)$.

Separating the variables, let us seek solution in the form:

$$u(x,y,t) = v(x)\, w(y)\, q(t). \qquad (2)$$

Substituting into (1), we have:

$$v(x)\,w(y)\,\frac{d^2q}{dt^2} - c^2 w(y)\,q(t)\,\frac{d^2v}{dx^2} - c^2 v(x)\,q(t)\,\frac{d^2w}{dy^2} = 0$$

Dividing by $v(x)\,w(y)\,q(t)$, we get:

$$\underbrace{\frac{1}{q}q''}_{\text{depends on } t} - \underbrace{c^2\frac{1}{v}v''}_{\text{depends on } x} - \underbrace{c^2\frac{1}{w}w''}_{\text{depends on } y} = 0 \qquad (3)$$

Hence, each of the terms must be a constant.

Let $\frac{1}{q}q'' = -\omega^2$ (negative separation constant).

$$q'' + \omega^2 q = 0$$

The general solution is

$$q(t) = C_1 \cos\omega t + C_2 \sin\omega t$$

or $q(t) = A\cos(\omega t + \phi)$.

With this choice of separation constant, (3) becomes:

$$-\omega^2 - c^2\frac{v''}{v} - c^2\frac{w''}{w} = 0.$$

or $$c^2\frac{v''}{v} + c^2\frac{w''}{w} + \omega^2 = 0$$

or $$\frac{v''}{v} + \frac{w''}{w} + k^2 = 0, \text{ where } k = \frac{\omega}{c}.$$

Let $\frac{v''}{v} = -k_1^2$, $\frac{w''}{w} = -k_2^2$, We choose negative constants here. See below.

so that $$-k_1^2 - k_2^2 + k^2 = 0. \qquad (4)$$

Equation for $v(x)$:

$$v'' + k_1^2 v = 0$$

General solution: $v(x) = A\cos k_1x + B\sin k_1x$.

The boundary conditions $u(0,y,t) = u(a,y,t) = 0$ applied to (2) mean: $v(0) = v(a) = 0$.

Hence $A = 0$ and $\underline{v(x) = B\sin k_1 x}$, and

$$B \sin k_1 a = 0$$

$$\Rightarrow \quad k_1 a = n\pi, \qquad n = 1, 2, \ldots$$

$$\underline{k_1 = \frac{n\pi}{a}}$$

Similarly, the equation for $w(y)$ is

$$w'' + k_2 w = 0.$$

Its solution $w(y) = C\cos k_2 y + D\sin k_2 y$ satisfies the boundary conditions $u(x,0,t) = u(x,b,t) = 0$, i.e. $w(0) = w(b) = 0$, only if $C = 0$,

$w(y) = D \sin k_2 y$, and $\sin k_2 b = 0$,

hence $k_2 b = m\pi, \quad m = 1, 2, \ldots$

$$\underline{k_2 = \frac{m\pi}{b}}.$$

[ Note that if we chose positive separation constants, e.g. $\frac{v''}{v} = k_1^2$, i.e. $v'' - k_1^2 v = 0$, the solution would be $v(x) = Ae^{k_1 x} + Be^{-k_1 x}$, and it would not satisfy $v(0) = v(a) = 0$ for nonzero $A$ or $B$, for any $k_1$.]

Substituting $k_1$ and $k_2$ into (4): $k^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$, and the frequency of harmonic vibrations is

$$\omega = ck = \pi c\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \equiv \omega_{nm}.$$

The corresponding solution is obtained by combining $q(t)$, $v(x)$ and $w(y)$ derived above into (2):

$$u(x,y,t) = \underset{\text{arbitrary constant}}{A}\ \underbrace{\sin\frac{n\pi x}{a}}_{v(x)}\ \underbrace{\cos\frac{m\pi y}{b}}_{w(y)}\ \underbrace{\cos(\omega_{nm} t + \phi)}_{q(t)}.$$