Partial Differential Equations AMA3006 — Problem sheet 3

Fourier series.

For a function $f(x)$, which is piecewise smooth in the interval $-\pi \le x \le \pi$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \tag{2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \tag{3}$$

holds for all x where $f(x)$ is continuous. If $f(x)$ is discontinuous at x, then the Fourier series on the right-hand side of Eq. (1) converges to $\frac{1}{2}[f(x-0) + f(x+0)]$.

Examples

1. Expand in the Fourier series the following functions $f(x)$ defined in the interval $(-\pi, \pi)$:

   (a) $f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0, \\ \frac{\pi}{4}, & 0 < x < \pi, \end{cases}$ (b) $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi, \end{cases}$ (c) $f(x) = |x|$, $|x| \le \pi$.

   In part (c), examine the answer for $x = 0$.

2. (a) Assuming that α is not an integer, find the Fourier series on $-\pi \le x \le \pi$ of the function $f(x) = \cos \alpha x$.

   (b) By setting $x = 0$ in the answer to part (a) show that for nonintegral α

   $$\frac{\pi}{\sin \pi\alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{\alpha + m} + \frac{1}{\alpha - m} \right).$$

   (c) By setting $x = \pi$ in the answer to part (a) show that for nonintegral α

   $$\pi \cot \pi\alpha = \frac{1}{\alpha} + \sum_{m=1}^{\infty} \left(\frac{1}{\alpha + m} + \frac{1}{\alpha - m} \right).$$

   Using this formula for a particular value of α, find an expression for π.

Homework problems

1. Expand in the Fourier series the following functions $f(x)$ defined in the interval $(-\pi, \pi)$:

   (a) $f(x) = x$, $-\pi < x < \pi$,

   (b) $f(x) = x^2$, $-\pi \le x \le \pi$,

   (c) $f(x) = \sin \alpha x$ for $-\pi < x < \pi$, assuming that α is not an integer.

2. Using the answer to question 1(b) for $x = 0$ and $x = \pi$, find the sums

   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad \text{and} \quad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$$

3. Show that the Fourier series for the function $f(x) = |\sin x|$ on $-\pi \le x \le \pi$ is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.$$

Using this result for a particular value of x, deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

4. Obtain the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ \sin x, & 0 \le x \le \pi. \end{cases}$$

Hence, deduce the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1}.$$

5. Obtain the half-range Fourier sine series of the function $f(x) = x(\pi - x)$ on $0 \le x \le \pi$.

[This is equivalent to assuming that $f(x)$ extends to $-\pi \le x \le 0$ as an odd function, i.e., $f(x) = x(\pi + x)$ here.]

To what value does the series converge for $x = \pi/2$? Deduce the value of the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}.$$