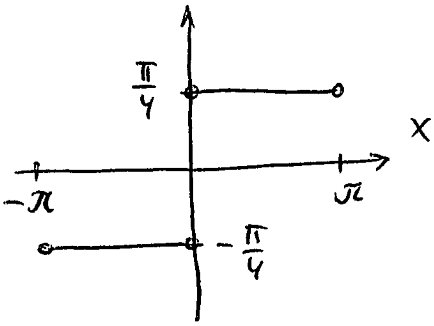
AMA3006 Problem sheet 3

Examples

1 (a)

$$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0 \\ \frac{\pi}{4}, & 0 < x < \pi \end{cases}$$



This is an odd function, hence all $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Both $f(x)$ and $\sin nx$ are odd functions; their product is even, hence we can replace $\int_{-\pi}^{\pi} \longrightarrow 2\int_0^{\pi}$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

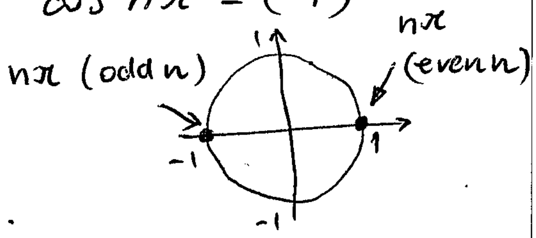
$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\sin nx\,dx$$

$$= \frac{1}{2}\int_0^{\pi}\sin nx\,dx = \frac{1}{2}\,\frac{1}{n}\left(-\cos nx\right)\Big|_0^{\pi}$$

$$= \frac{1}{2n}\left(-\cos n\pi + 1\right)$$

$$= \frac{1}{2n}\left(1-(-1)^n\right) = \begin{cases} 0, & n \text{ even} \\ \frac{1}{n}, & n \text{ odd.} \end{cases}$$

$\cos n\pi = (-1)^n$



Hence, only odd n contribute.
We can write a generic odd number as $n = 2m+1$ $(m = 0, 1, \ldots)$

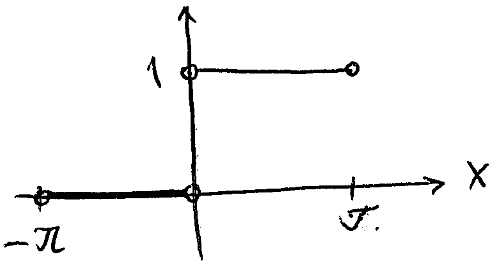
Hence, the Fourier series sought is:

$$\sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1}.$$

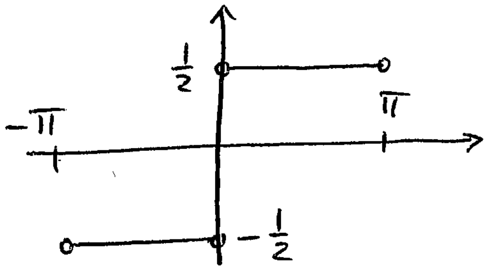
Note that taking $x = \frac{\pi}{2}$, $\sin(2m+1)\frac{\pi}{2} = \sin(m\pi + \frac{\pi}{2}) = (-1)^m$, we prove:
$$\frac{\pi}{4} = \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}$$

<u>Note:</u> For $x = 0$ the series gives 0,
which is $\frac{1}{2}\left[f(0-0)+f(0+0)\right] = \frac{1}{2}\left[-\frac{\pi}{4}+\frac{\pi}{4}\right] = 0$,
as expected.

(b) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$



Note: at $x = 0$ this function has a discontinuity (jump) up, equal to 1.
Compare with $f(x)$ from (a), which had a jump (similar) equal to $\frac{\pi}{2}$. If we multiply $f(x)$ from (a) by $\frac{2}{\pi}$ it will become:

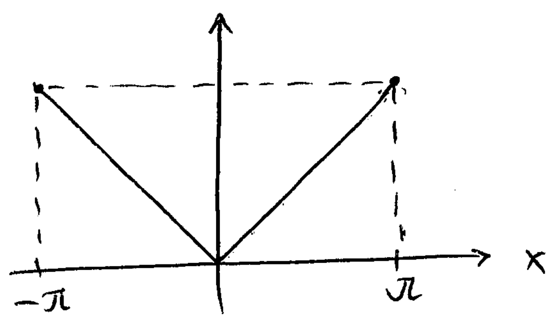


By adding $\frac{1}{2}$ we can obtain the above $f(x)$. Hence, the Fourier series can be obtained by taking the result of 1(a), multiplying by $\frac{2}{\pi}$ and adding $\frac{1}{2}$. Hence, we have the Fourier series:

$$\frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1} .$$

Obviously, for $x = 0$ this gives $\frac{1}{2}$, which is exactly $\frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{1}{2}[0+1] = \frac{1}{2}$.

(c) $f(x) = |x| \quad -\pi \le x \le \pi$



This is an <u>even</u> function, hence only a_n are nonzero, and only a_0 and the cosine terms will be present.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \, dx$$

since the integrand is an even function.

$n=0$:
$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\,\frac{x^2}{2}\Big|_0^\pi = \frac{\cancel{2}\,\pi^{\cancel{2}}}{\cancel{\pi}\cdot\cancel{2}} = \pi .$$

$n>0$:
$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$
$$= \frac{2}{\pi}\,\frac{1}{n}\int_0^\pi x\,d(\sin nx)$$

We use integration by parts here

$\cos nx\,dx = \frac{1}{n}\,d(\sin nx)$

$$= \underbrace{\frac{2}{\pi}\cdot\frac{1}{n}\,x\sin nx\Big|_0^\pi}_{=\,0} - \frac{}{\pi n}\int_0^\pi \sin nx\,dx$$

($\sin n\pi = 0$, $\sin 0 = 0$)

$$= -\frac{2}{\pi n^2}\left(-\cos nx\right)\Big|_0^\pi = -\frac{2(1-\cos n\pi)}{\pi n^2}$$

$$\cos n\pi = (-1)^n \quad\Rightarrow\quad a_n = -\frac{2(1-(-1)^n)}{\pi n^2} = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4}{\pi n^2}, & n \text{ odd}. \end{cases}$$

Using $2m+1$ as a generic odd number, we have the following Fourier series:

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2} .$$

Take $x=0$. Then $f(x)=0$. Since $f(x)$ is continuous at $x=0$, the series should have the same value. Using $\cos(2m+1)\cdot 0 = \cos 0 = 1$, we then have:

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}, \quad \text{i.e.} \quad \underbrace{1+\frac{1}{3^2}+\frac{1}{5^2}}_{\approx 1.151\ldots}+\ldots = \underbrace{\frac{\pi^2}{8}}_{\approx 1.2337} .$$

Check: for just three terms:

② (a) $f(x) = \cos\alpha x$ is an even function;
hence $b_n = 0$, and we can find a_n as:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\alpha x \cos nx \, dx$$

For $n=0$: $$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos\alpha x \, dx = \frac{2}{\alpha\pi} \sin\alpha x \Big/_0^{\pi}$$

$$\Rightarrow \quad a_0 = \frac{2\sin\alpha\pi}{\alpha\pi}.$$

<u>$n>0$:</u>

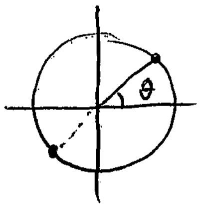
$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\alpha x \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(\alpha+n)x + \cos(\alpha-n)x\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\cos(\alpha+n)x + \cos(\alpha-n)x\right] dx$$

$$= \frac{1}{\pi}\left\{ \frac{1}{\alpha+n}\sin(\alpha+n)x\Big|_0^{\pi} + \frac{1}{\alpha-n}\sin(\alpha-n)x\Big|_0^{\pi} \right\}$$

$\sin(\theta+n\pi)$
$= (-1)^n \sin\theta$



$$= \frac{1}{\pi}\left[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n \sin\alpha\pi}{\alpha+n} + \frac{(-1)^n \sin\alpha\pi}{\alpha-n}\right]$$

$$= (-1)^n \frac{\sin\alpha\pi}{\pi}\left(\frac{1}{\alpha+n} + \frac{1}{\alpha-n}\right)$$

Hence the Fourier series is:

$$\cos\alpha x = \frac{\sin\alpha\pi}{\alpha\pi} + \frac{\sin\alpha\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha+n} + \frac{1}{\alpha-n}\right)\cos nx$$

or $$\cos\alpha x = \frac{\sin\alpha\pi}{\pi}\left[\frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha+n} + \frac{1}{\alpha-n}\right)\cos nx\right].$$

(b) Setting $x=0$ in the last equation of part (a), we have:

$$1 = \frac{\sin\alpha\pi}{\pi}\left[\frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha+n}+\frac{1}{\alpha-n}\right)\right]$$

or $$\frac{\pi}{\sin\alpha\pi} = \frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha+n}+\frac{1}{\alpha-n}\right),$$ as required.

(c) Setting $x=\pi$ gives:

$$\cos\alpha\pi = \frac{\sin\alpha\pi}{\pi}\left[\frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha+n}+\frac{1}{\alpha-n}\right)\underbrace{\cos n\pi}_{=(-1)^n}\right]$$

$$\frac{\pi\cos\alpha\pi}{\sin\alpha\pi} = \frac{1}{\alpha} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha+n}+\frac{1}{\alpha-n}\right)$$

$$\underline{\pi\cot\alpha\pi = \frac{1}{\alpha} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha+n}+\frac{1}{\alpha-n}\right)},$$ as required.

Note that this formula immetialy shows that $\cot\theta\to\infty$ for $\theta\to m\pi$ ($m\in\mathbb{Z}$), as in this case $\alpha\to m$, and one of the denominators on the right-hand side tends to zero.

Take $\alpha=\frac{1}{4}$ in the last equation.

$$\underbrace{\pi\cot\frac{\pi}{4}}_{=1} = 4 + \sum_{n=1}^{\infty}\left(\frac{1}{\frac{1}{4}+n}+\frac{1}{\frac{1}{4}-n}\right)$$

$$\pi = 4 + \sum_{n=1}^{\infty}\frac{\frac{1}{4}-n+\frac{1}{4}+n}{(\frac{1}{4}+n)(\frac{1}{4}-n)}$$

$$\pi = 4 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\frac{1}{16}-n^2}$$

or $$\boxed{\pi = 4 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2-1/16}}$$

Numerically:

$$4-\frac{1}{2}\sum_{n=1}^{100}\frac{1}{n^2-1/16}\approx 3.14657$$

$$4-\frac{1}{2}\sum_{n=1}^{1000}\frac{1}{n^2-1/16}\approx 3.14209$$