AMA 3006 Problem sheet 4

Examples

① We need to solve the wave equation for the string,

$$u_{tt} - c^2 u_{xx} = 0 \qquad (1)$$

for $u(x,t)$, $0 \le x \le \ell$, subject to the boundary conditions,

$$u(0,t) = u(\ell,t) = 0. \qquad (2)$$

and initial conditions:

$$u(x,0) = 0, \qquad (3)$$

$$u_t(x,0) = \begin{cases} 0, & 0 < x < \frac{\ell}{2} - \delta \\ v_0, & \frac{\ell}{2} - \delta < x < \frac{\ell}{2} + \delta \\ 0, & \frac{\ell}{2} + \delta < x < \ell \end{cases} \qquad (4)$$

Solving (1) by the variable separation method, we seek solution in the form

$$u(x,t) = v(x)\, q(t)$$

and obtain (see Ch. 2 and Problem sheet 2):

$$u(x,t) = \underbrace{(A \cos \omega_n t + B \sin \omega_n t)}_{q(t)} \underbrace{\sin \frac{\pi n x}{\ell}}_{v(x)}, \qquad n = 1, 2, \ldots$$

which satisfies boundary conditions (2), and where

$$\omega_n = \frac{\pi n c}{\ell}.$$

Using the superposition principle, we can construct a more general solution:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos\omega_n t + B_n \sin\omega_n t) \sin\frac{n\pi x}{l},$$

where A_n and B_n are arbitrary coefficients. We find them from the initial conditions.

Using (3), set $t=0$ in $u(x,t)$:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{l} \overset{\text{By (3)}}{=} 0 \Rightarrow A_n = 0$$

for all n.

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\omega_n t \sin\frac{n\pi x}{l}. \qquad (5)$$

Taking the time derivative:

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \omega_n \cos\omega_n t \sin\frac{n\pi x}{l}$$

Setting $t=0$:

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \omega_n \sin\frac{n\pi x}{l}.$$

This is a Fourier sine series with coefficients $B_n \omega_n$. It must satisfy condition (4). Hence:

$$B_n \omega_n = \frac{2}{l}\int_0^l u_t(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_{\frac{l}{2}-\delta}^{\frac{l}{2}+\delta} v_0 \sin\frac{n\pi x}{l}\,dx$$

only the interval $(\frac{l}{2}-\delta, \frac{l}{2}+\delta)$ contributes to the integral

$$= \frac{2v_0}{l}\,\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)\Bigg|_{\frac{l}{2}-\delta}^{\frac{l}{2}+\delta}$$

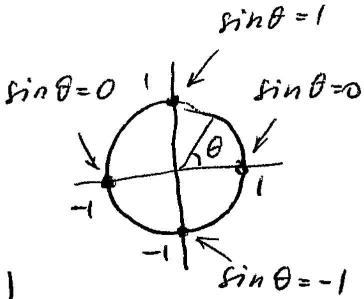
$$= \frac{2v_0}{\pi n}\left[-\cos\frac{n\pi}{\ell}\left(\frac{\ell}{2}+\delta\right)+\cos\frac{n\pi}{\ell}\left(\frac{\ell}{2}-\delta\right)\right]$$

$$= \frac{2v_0}{\pi n}\left[-\cos\left(\frac{n\pi}{2}+\frac{n\pi\delta}{\ell}\right)+\cos\left(\frac{n\pi}{2}-\frac{n\pi\delta}{\ell}\right)\right]$$

$$= \frac{2v_0}{\pi n}\left[\cancel{-\cos\frac{n\pi}{2}\cos\frac{n\pi\delta}{\ell}}+\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell}+\cancel{\cos\frac{n\pi}{2}\cos\frac{n\pi\delta}{\ell}}\right.$$

$$\left.+\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell}\right]$$

$$= \frac{4v_0}{\pi n}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{\ell}$$



Here $$\sin\frac{n\pi}{2} = \begin{cases} 0, & n=2m \text{ (even)} \\ (-1)^m, & n=2m+1 \text{ (odd)} \end{cases}$$
$m = 0, 1, 2, \ldots$

Hence we have nonzero coefficients for odd $n=2m+1$

$$B_n\omega_n = \frac{4v_0}{\pi(2m+1)}(-1)^m\sin\frac{(2m+1)\pi\delta}{\ell}$$

Substituting $\omega_n = \dfrac{\pi(2m+1)c}{\ell}$, we find:

$$\underset{(n=2m+1)}{B_n} = \frac{4v_0\ell}{\pi^2 c}\,\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi\delta}{\ell}.$$

Hence, the solution, Eq. (5), is:

$$u(x,t) = \frac{4v_0\ell}{\pi^2 c}\sum_{m=0}^{\infty}\frac{(-1)^m\sin\frac{(2m+1)\pi\delta}{\ell}}{(2m+1)^2}\sin\frac{\pi(2m+1)c}{\ell}t$$
$$\times\sin\frac{(2m+1)\pi x}{\ell}$$

Now, the momentum imparted to the string is related to the velocity v_0 by: $P = \underbrace{\rho\,2\delta}_{\text{mass of string of length } 2\delta}\,v_0$,
where ρ is the linear mass density.

We can thus, substitute $v_0 = \frac{P}{2\rho\delta}$ into $u(x,t)$ and take the limit $\delta \to 0$ (string hit at a point).

In this case $\sin\frac{(2m+1)\pi\delta}{l} \simeq \frac{(2m+1)\pi\delta}{l}$,

($\sin x \simeq x$ for $|x| \ll 1$)

and cancelling δ and some other factors, we obtain:

$$u(x,t) = \frac{2P}{\pi\rho c}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\sin\frac{(2m+1)\pi c}{l}t\ \sin\frac{(2m+1)\pi x}{l}.$$

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(2) Solve the heat equation for the rod,

$$u_t - K u_{xx} = 0, \qquad (1)$$

for $u(x,t)$, $0 \le x \le l$, with insulating boundary conditions

$$u_x(0,t) = u_x(l,t) = 0 \qquad (2)$$

and initial condition:

$$u(x,0) = f(x). \qquad (3)$$

Recall: $j = -\varkappa\frac{\partial u}{\partial x}$. Hence, zero heat flux density j means $\frac{\partial u}{\partial x} = 0$

Using variable separation for (1), we seek its solution in the form:

$$u(x,t) = v(x)\,q(t), \qquad (4)$$

and have:

$$v(x)\frac{dq}{dt} - q(t)\,K\frac{d^2v}{dx^2} = 0$$

Dividing through by $v(x)q(t)$,

$$\underbrace{\frac{1}{q(t)}\frac{dq}{dt}}_{\text{const } -\lambda} - \underbrace{\frac{K}{v(x)}\frac{d^2v}{dx^2}}_{-\lambda} = 0$$

Denoting the separation constant $-\lambda$, we have:

$$\frac{1}{q(t)}\frac{dq}{dt} = -\lambda$$

$$\frac{dq}{dt} = -\lambda q ,$$

which has the solution

$$q(t) = C e^{-\lambda t} , \qquad C \text{ - arbitrary constant}$$

The coordinate part gives:

$$\frac{K}{v(x)}\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} + \frac{\lambda}{K} v = 0$$

The general solution:

$$v = A\cos\sqrt{\frac{\lambda}{K}}x + B\sin\sqrt{\frac{\lambda}{K}}x .$$

$U_x(x,t) = \frac{dv}{dx} q(t)$, so boundary condition (2) means that $v'(0) = v'(\ell) = 0$.

$$\frac{dv}{dx} = -A\sqrt{\frac{\lambda}{K}}\sin\sqrt{\frac{\lambda}{K}}x + B\lambda\cos\sqrt{\frac{\lambda}{K}}x .$$

$x = 0$: $\Rightarrow B\lambda = 0 \Rightarrow B = 0$

$$\Rightarrow v(x) = A\cos\sqrt{\frac{\lambda}{K}}x .$$

$x = \ell$: $-A\sqrt{\frac{\lambda}{K}}\sin\sqrt{\frac{\lambda}{K}}\ell = 0 \Rightarrow \sqrt{\frac{\lambda}{K}}\ell = n\pi, \; n = 0, 1, \ldots$

So, $$\sqrt{\frac{\lambda}{K}} = \frac{n\pi}{l}$$

$$\lambda = \frac{n^2\pi^2}{l^2} K , \qquad n = 0, 1, 2, \dots$$

and (4) gives:

$$u(x,t) = A \cos \frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{l^2}Kt} .$$

Using the superposition principle, we construct a more general solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{l^2}Kt} . \qquad (5)$$

This must satisfy the initial condition. Setting $t=0$,

$$\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} = f(x) .$$

So, A_n are the coefficients of the cosine Fourier expansion. Its standard form is in fact

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} = f(x),$$

and the coefficients are given by:

$$A_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$A_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

Here x is the integration variable, and it can be denoted by any symbol, e.g. ξ, to avoid confusion in the final formula.

Using these, we obtain from (5) (where we again single out the $n=0$ term):

$$u(x,t) = \frac{1}{l}\int_0^l f(\xi)\, d\xi + \frac{2}{l}\sum_{n=1}^{\infty}\left(\int_0^l f(\xi) \cos\frac{n\pi\xi}{l} d\xi\right) \cos \frac{n\pi x}{l} \times e^{-\frac{n^2\pi^2}{l^2}Kt} .$$

It is interesting to note that all terms in the sum $\sum_{n=1}^{\infty}$ tend to zero as $t \to +\infty$,

since

$$e^{-\frac{n^2\pi^2}{\ell^2}Kt} \xrightarrow[t\to+\infty]{} 0 .$$

In the large-time limit $u(x,t)$ becomes a constant:

$$u(x,t) = \frac{1}{\ell}\int_0^{\ell} f(\xi)\, d\xi \qquad (*) .$$

It is easy to recognise the right-hand side as the average initial temperature in the rod. Since the ends of the rod are insulated, the heat energy (which is proportional to the temperature u) distributes uniformly along the rod as the times goes by, leading to the long-time equilibrium $(*)$.

3 Solve

$$u_{tt} - c^2 u_{xx} = \frac{q}{\rho} \qquad (1)$$

with boundary conditions for $u(x,t)$ on $0 \le x \le \ell$:

$$u(0,t) = u(\ell,t) = 0 \qquad (2)$$

and initial conditions

$$u(x,0) = 0 \qquad (3)$$

$$u_t(x,0) = 0 , \qquad (4)$$

i.e. the string is at equilibrium at $t=0$, just before the load (right-hand side in Eq. (1)) is applied.

Let us seek the solution in the form

$$u(x,t) = \tilde{u}(x,t) + u_s(x), \qquad (5)$$

where $u_s(x)$ is the time-independent solution of (1):

$$-c^2 (u_s)_{xx} = \frac{q}{\rho} \qquad (6)$$

$$-c^2 \frac{d^2 u_s}{dx^2} = \frac{q}{\rho}$$

$$\frac{d^2 u_s}{dx^2} = -\frac{q}{c^2 \rho}$$

Integrating: $$\frac{du_s}{dx} = -\frac{qx}{c^2\rho} + C_1$$

and again: $$u_s = -\frac{qx^2}{2c^2\rho} + C_1 x + C_2,$$

where C_1 and C_2 can be found by using the boundary conditions: $u_s(0) = u_s(l) = 0$:

Hence: $$C_2 = 0$$

$$-\frac{ql^2}{2c^2\rho} + C_1 l = 0$$

$$C_1 = \frac{ql}{2c^2\rho}$$

Hence: $$u_s = -\frac{qx^2}{2c^2\rho} + \frac{ql}{2c^2\rho}x,$$

or $$\underline{u_s(x) = \frac{q}{2c^2\rho}x(l-x)}. \qquad (7)$$

Substituting (5) into (1) we have:

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} - c^2 \frac{\partial^2 u_s}{\partial x^2} = \frac{q}{\rho},$$

and because of (6):

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = 0,$$

so that $\tilde{u}(x,t)$ satisfies the homogeneous wave equation.

Using variable separation and boundary conditions (9
(2), and the superposition principle, we construct the following solution (see Ch. 2, also Q.1 above):

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{\ell},$$

where $\omega_n = \frac{n\pi c}{\ell}$.

This gives, by combining $\tilde{u}(x,t)$ with Eq. (7):

$$u(x,t) = \frac{q}{2c^2\rho} x(\ell - x) + \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{\ell}.$$

This must satisfy the initial conditions:
From (3), setting $t=0$ in the above expression,

$$\frac{q}{2c^2\rho} x(\ell - x) + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = 0 \tag{8}$$

Taking the time derivative of $u(x,t)$ and setting $t=0$, using (4), we have:

$$\sum_{n=1}^{\infty} B_n \omega_n \sin \frac{n\pi x}{\ell} = 0. \tag{9}$$

From (9), $B_n = 0$ for all n.

From (8):

$$-\frac{q}{2c^2\rho} x(\ell - x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}.$$

The right-hand side is a sine Fourier series on $(0, \ell)$ of the function on the left. Hence, A_n are found using the standard formulae:

$$A_n = \frac{2}{\ell} \int_0^{\ell} \left(-\frac{q}{2c^2\rho} x(\ell - x) \right) \sin \frac{n\pi x}{\ell} \, dx.$$

Carrying out the integration,

$$A_n = -\frac{q}{c^2\rho l}\underbrace{\int_0^l (xl - x^2)\sin\frac{n\pi x}{l}\,dx}$$

$$= l\int_0^l x\sin\frac{n\pi x}{l}\,dx - \int_0^l x^2\sin\frac{n\pi x}{l}\,dx$$

$$= l\,x\,\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)\Big|_0^l + \frac{l^2}{n\pi}\int_0^l \cos\frac{n\pi x}{l}\,dx$$

$$- x^2\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)\Big|_0^l - \frac{2l}{n\pi}\int_0^l \cos\frac{n\pi x}{l}\,x\,dx$$

$$= \cancel{-\frac{l^3}{n\pi}\cos n\pi} + \frac{l^2}{n\pi}\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_0^l \qquad \left.\begin{array}{l}\sin n\pi = 0\\ \sin 0 = 0\end{array}\right\}$$

$$\cancel{+\frac{l^3}{n\pi}\cos n\pi} - \frac{2l}{n\pi}x\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_0^l + \frac{2l^2}{n^2\pi^2}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2l^2}{n^2\pi^2}\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)\Big|_0^l \qquad \cos n\pi = (-1)^n$$

$$= \frac{2l^3}{n^3\pi^3}\left(-\cos n\pi + 1\right)$$

$$= \frac{2l^3}{n^3\pi^3}\underbrace{\left(1-(-1)^n\right)}_{\substack{0 \text{ for even } n\\ 2 \text{ for odd } n}} = \begin{cases} 0, & n = 2m \\ \dfrac{4l^3}{n^3\pi^3}, & n = 2m+1 \end{cases}, \quad m = 0, 1, \ldots$$

Hence

$$A_n = -\frac{4l^3 q}{c^2\rho l\, n^3\pi^3}, \quad n = 2m+1$$

$$= -\frac{4ql^2}{c^2\rho\pi^3}\frac{1}{(2m+1)^3},$$

and the solution is:

$$u(x,t) = \frac{q}{2c^2\rho}\left[x(l-x) - \frac{8l^2}{\pi^3}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\cos\frac{(2m+1)\pi c}{l}t\,\sin\frac{(2m+1)\pi x}{l}\right]$$

(here we substituted A_n and $B_n(=0)$ into $u(x,t)$ from page 9).