AMA3006 Problem sheet 4

Homework problems

SOLUTIONS

(1) Solve $u_t - K u_{xx} = 0$ for $u(x,t)$, $0 \le x \le \ell$, with boundary conditions $u(0,t) = u(\ell,t) = 0$, and initial condition $u(x,0) = T$.

Using variable separation, seek solution in the form:

$$u(x,t) = v(x)\, q(t).$$

Substituting into the heat equation, we have:

$$v(x)\frac{dq}{dt} - q(t)\, K \frac{d^2 v}{dx^2} = 0.$$

Dividing by $v(x)\, q(t)$:

$$\underbrace{\frac{1}{q(t)}\frac{dq}{dt}}_{-\lambda} - \underbrace{K\frac{1}{v(x)}\frac{d^2 v}{dx^2}}_{-\lambda} = 0$$

Solving the time equation,

$$\frac{dq}{dt} = -\lambda q,$$

we obtain $q(t) = C e^{-\lambda t}$.

The coordinate equation:

$$\frac{K}{v(x)} v''(x) = -\lambda,$$

$$v''(x) + \frac{\lambda}{K} v(x) = 0.$$

The general solution of this equation is

$$v(x) = A\cos\sqrt{\frac{\lambda}{K}}\, x + B\sin\sqrt{\frac{\lambda}{K}}\, x.$$

To satisfy the boundary conditions, we require $v(0) = v(\ell) = 0$.

For $x = 0$, we have $A = 0$,

hence $v(x) = B \sin \sqrt{\frac{\lambda}{K}}\, x$.

For $x = \ell$:

$$B \sin \sqrt{\frac{\lambda}{K}}\, \ell = 0,$$

so that
$$\sqrt{\frac{\lambda}{K}}\, \ell = n\pi, \qquad n = 1, 2, \dots$$

$$\sqrt{\frac{\lambda}{K}} = \frac{n\pi}{\ell}$$

$$\lambda = \frac{n^2\pi^2}{\ell^2} K.$$

Hence:

$$u(x,t) = B_n \sin \frac{n\pi x}{\ell}\, e^{-\frac{n^2\pi^2}{\ell^2}Kt},$$

where B_n is an arbitrary constant.

Combining these solutions using the superposition principle,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell}\, e^{-\frac{n^2\pi^2}{\ell^2}Kt} \qquad (*)$$

To satisfy the initial conditions at $t = 0$, we require

$$T = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell}.$$

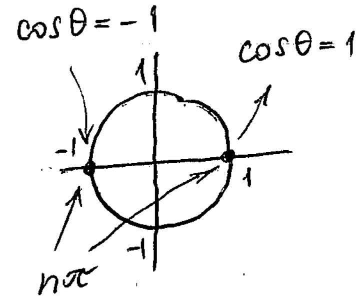
Finding B_n:

$$B_n = \frac{2}{\ell} \int_0^{\ell} T \sin \frac{n\pi x}{\ell}\, dx$$

$$= \frac{2T}{\ell} \frac{\ell}{n\pi} \left(-\cos \frac{n\pi x}{\ell}\right) \Big|_0^{\ell}$$

$$= \frac{2T}{n\pi} (\underbrace{-\cos n\pi}_{(-1)^n} + 1)$$

$$= \frac{2T}{n\pi} \left(1 - (-1)^n\right)$$



Hence:

$$B_n = \begin{cases} 0, & n = 2m \quad (\text{even } n) \\ \dfrac{4T}{n\pi}, & n = 2m+1 \quad (\text{odd } n) \end{cases}$$

$$m = 0, 1, 2, \ldots$$

Using this in (*), we obtain the final answer:

$$u(x,t) = \frac{4T}{\pi} \sum_{m=0}^{\infty} \frac{\sin \frac{(2m+1)\pi x}{\ell}}{2m+1} e^{-\frac{(2m+1)^2\pi^2}{\ell^2}Kt}.$$

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(2) Solving the same problem as (1) with a different initial condition, $u(x,0) = \frac{4T}{\ell^2}x(\ell-x)$, we repeat all steps that led to equation (*) (page 2). Imposing the initial condition:

$$\frac{4T}{\ell^2}x(\ell-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$B_n = \frac{2}{\ell}\int_0^\ell \frac{4T}{\ell^2}x(\ell-x)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{8T}{\ell^3}\left[\ell\int_0^\ell x\sin\frac{n\pi x}{\ell}\,dx - \int_0^\ell x^2\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$= \frac{8T}{\ell^3}\left[\ell x\frac{\ell}{n\pi}\left(-\cos\frac{n\pi x}{\ell}\right)\Big|_0^\ell + \frac{\ell^2}{n\pi}\int_0^\ell \cos\frac{n\pi x}{\ell}\,dx\right.$$

$$\left. + x^2\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_0^\ell - \frac{2\ell}{n\pi}\int_0^\ell \cos\frac{n\pi x}{\ell}\,x\,dx\right]$$

Integrating by parts

$$= \frac{8T}{\ell^3}\left[\cancel{-\frac{\ell^3}{n\pi}\cos n\pi} + \frac{\ell^2}{n\pi}\cdot\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^\ell\right.$$

$$\left. + \cancel{\frac{\ell^3}{n\pi}\cos n\pi} - \frac{2\ell}{n\pi}\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^\ell + \frac{2\ell^2}{n^2\pi^2}\int_0^\ell \sin\frac{n\pi x}{\ell}\,dx\right]$$

$\sin n\pi = 0$
$\sin 0 = 0$

$$= \frac{8T}{\ell^3}\left[\ \frac{2\ell^2}{n^2\pi^2}\ \frac{\ell}{n\pi}\left(-\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^\ell\ \right]$$

$$= \frac{16T}{n^3\pi^3}\left(1-(-1)^n\right) = \begin{cases} 0, & n = 2m \\ \dfrac{32T}{n^3\pi^3}, & n = 2m+1, \end{cases} \quad m = 0, 1, 2, \ldots$$

Substituting this into (*) we obtain the answer:

$$u(x,t) = \frac{32T}{\pi^3}\sum_{m=0}^{\infty}\frac{\sin\frac{(2m+1)\pi x}{\ell}}{(2m+1)^3}\, e^{-\frac{(2m+1)^2\pi^2}{\ell^2}Kt}.$$

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3 Heat equation: $u_t - Ku_{xx} = 0, \qquad 0 \le x \le \ell$

boundary conditions: $u_x(0,t) = u_x(\ell,t) = 0,$

initial condition: $u(x,0) = \frac{4T}{\ell^2}x(\ell - x).$

Using variable separation:

$$u(x,t) = v(x)\,q(t),$$

$$v(x)\frac{dq}{dt} - q(t)\,K\frac{d^2v}{dx^2} = 0$$

$$\underbrace{\frac{1}{q(t)}\frac{dq}{dt}}_{-\lambda} - \underbrace{\frac{K}{v(x)}\frac{d^2v}{dx^2}}_{-\lambda} = 0$$

Time equation $\frac{dq}{dt} = -\lambda q \Rightarrow q(t) = Ce^{-\lambda t}.$

Coordinate equation:

$$\frac{d^2v}{dx^2} + \frac{\lambda}{K}v(x) = 0$$

$$\Rightarrow \quad v(x) = A\cos\sqrt{\frac{\lambda}{K}}\,x + B\sin\sqrt{\frac{\lambda}{K}}\,x.$$

Boundary condition on $u_x(x,t) = v'(x)\,q(t),$

requires $\quad v'(0)=0, \quad v'(\ell)=0.$

$$v'(x) = -A\sqrt{\frac{\lambda}{K}}\sin\sqrt{\frac{\lambda}{K}}x + B\sqrt{\frac{\lambda}{K}}\cos\sqrt{\frac{\lambda}{K}}x .$$

$x=0$: $\quad B\sqrt{\frac{\lambda}{K}}\cdot 1 = 0 \Rightarrow B=0$ (or $\lambda=0$ – see below)

$$\Rightarrow \quad v(x) = A\cos\sqrt{\frac{\lambda}{K}}x .$$

$x=0$: $\quad v'(\ell)=0$ gives $-A\sqrt{\frac{\lambda}{K}}\sin\sqrt{\frac{\lambda}{K}}\ell = 0$

$$\Rightarrow \quad \sqrt{\frac{\lambda}{K}}\ell = n\pi, \quad n = 0,1,2,\dots$$

$$\sqrt{\frac{\lambda}{K}} = \frac{n\pi}{\ell}$$

$$\lambda = \frac{n^2\pi^2}{\ell^2}K$$

Hence:
$$u(x,t) = A_n\cos\frac{n\pi x}{\ell}\, e^{-\frac{n^2\pi^2}{\ell^2}Kt},$$
where A_n are arbitrary constants.

Using the superposition principle, we construct:

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{\ell}\, e^{-\frac{n^2\pi^2}{\ell^2}Kt}. \tag{1}$$

This solution must satisfy the initial condition.

Setting $t=0$:

$$\frac{4T}{\ell^2}x(\ell-x) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{\ell}. \tag{2}$$

The right-hand side is a cosine Fourier series, and traditionally, the $n=0$ term is written out separtately, so that (1) becomes (writing $\frac{A_0}{2}$ instead of A_0)

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell}\, e^{-\frac{n^2\pi^2}{\ell^2}Kt}, \tag{3}$$

and (2):
$$\frac{4T}{\ell^2}x(\ell-x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell}.$$

Let's find the coefficients.

$$A_0 = \frac{2}{\ell}\int_0^\ell \frac{4T}{\ell^2} x(\ell-x)\,dx$$

$$= \frac{8T}{\ell^3}\int_0^\ell (\ell x - x^2)\,dx = \frac{8T}{\ell^3}\left[\ell\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\ell$$

$$= \frac{8T}{\ell^3}\left[\frac{\ell^3}{2} - \frac{\ell^3}{3}\right] = \frac{8T}{6} \Rightarrow A_0 = \frac{4T}{3}.$$

$$A_n = \frac{2}{\ell}\int_0^\ell \frac{4T}{\ell^2} x(\ell-x)\cos\frac{n\pi x}{\ell}\,dx$$

$$= \frac{8T}{\ell^3}\left[\ell\int_0^\ell x\cos\frac{n\pi x}{\ell}\,dx - \int_0^\ell x^2\cos\frac{n\pi x}{\ell}\,dx\right]$$

$$= \frac{8T}{\ell^3}\left[\underbrace{\ell x\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^\ell}_{0=} - \frac{\ell^2}{n\pi}\int_0^\ell \sin\frac{n\pi x}{\ell}\,dx\right.$$

$$\left. - \underbrace{x^2\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^\ell}_{=0} + \frac{2\ell}{n\pi}\int_0^\ell \sin\frac{n\pi x}{\ell}\,x\,dx\right]$$

$$= \frac{8T}{\ell^3}\left[+\frac{\ell^2}{n\pi}\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_0^\ell - \frac{2\ell}{n\pi}x\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_0^\ell\right.$$

$$\left. + \frac{2\ell^2}{n^2\pi^2}\int_0^\ell \cos\frac{n\pi x}{\ell}\,dx\right]$$

$$= \frac{8T}{\ell^3}\left[\frac{\ell^3}{n^2\pi^2}(\cos n\pi - 1) - \frac{2\ell^3}{n^2\pi^2}\cos n\pi\right.$$

$$\left. + \underbrace{\frac{2\ell^2}{n^2\pi^2}\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^\ell}_{=0}\right] \quad \left.\begin{aligned}\sin n\pi &= 0\\ \sin 0 &= 0.\end{aligned}\right\}$$

$$= -\frac{8T}{\ell^3}\cdot\frac{\ell^3}{n^2\pi^2}(1+\cos n\pi) = -\frac{8T}{n^2\pi^2}\left(1+(-1)^n\right)$$

$$= \begin{cases} 0, & n = 2m+1 \quad \text{(odd } n) \\ -\dfrac{16T}{n^2\pi^2}, & n = 2m, \quad m = 1, 2, \ldots \text{ (even } n). \end{cases}$$

Substituting A_0 and A_n into (3), we obtain the final answer:

$$u(x,t) = \frac{2T}{3} - \frac{16T}{\pi^2}\sum_{m=1}^{\infty} \frac{\cos\frac{2m\pi x}{\ell}}{4m^2} e^{-\frac{4m^2\pi^2}{\ell^2}Kt}.$$

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(4) The displacement of the string, $u(x,t)$, obeys the wave equation: $u_{tt} - c^2 u_{xx} = 0$.

Boundary conditions: $u(0,t) = u(\ell,t) = 0$.

Initial conditions: $u(x,0) = \frac{4h}{\ell^2}x(\ell - x)$,

$u_t(x,0) = 0$.

Using variable separation,

$$u(x,t) = v(x)\,q(t).$$

Substituting into the wave equation and dividing by $v(x)q(t)$:

$$\underbrace{\frac{1}{q(t)}\frac{d^2q}{dt^2}}_{\text{const} = -\omega^2} - c^2\underbrace{\frac{1}{v(x)}\frac{d^2v}{dx^2}}_{-\omega^2} = 0.$$

The equation for $q(t)$:

$$\frac{d^2q}{dt^2} + \omega^2 q(t) = 0.$$

General solution:

$$q(t) = A\cos\omega t + B\sin\omega t.$$

The coordinate equation:

$$\frac{d^2v}{dx^2} + \frac{\omega^2}{c^2}v(x) = 0.$$

Introducing $k=\frac{\omega}{c}$, we have:

$$v''+k^2 v=0.$$

The general solution of this equation reads

$$v(x)= C_1 \cos kx + C_2 \sin kx .$$

To satisfy the boundary conditions, we require

$$v(0)=0 \quad \Rightarrow \quad C_1=0$$

$$v(\ell)=0 \quad \Rightarrow \quad C_2 \sin k\ell = 0,$$

hence $\quad k\ell = n\pi, \qquad n=1,2,\dots$

$$k=\frac{n\pi}{\ell}$$

$$\omega = kc = \frac{n\pi c}{\ell} \equiv \omega_n .$$

Hence, we have obtained the following solution:

$$u(x,t) = C_2 \sin\frac{n\pi x}{\ell}\left(A\cos\omega_n t + B\sin\omega_n t\right).$$

In fact, there is no need to keep C_2, since A and B are arbitrary constants. Renaming them A_n and B_n, and using the superposition principle, we construct:

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\omega_n t + B_n \sin\omega_n t\right)\sin\frac{n\pi x}{\ell}. \quad (*)$$

To apply the initial conditions, we also need its time derivative,

$$u_t(x,t)=\sum_{n=1}^{\infty}\left(-A_n\omega_n\sin\omega_n t + B_n\omega_n\cos\omega_n t\right)\sin\frac{n\pi x}{\ell}.$$

Applying $u_t(x,0)=0$ first,

$$\sum_{n=1}^{\infty} B_n\omega_n \underbrace{\cos\omega_n t}_{\substack{\shortparallel\\ 1 \text{ for } t=0}} \sin\frac{n\pi x}{\ell} = \sum_{n=1}^{\infty} B_n\omega_n \sin\frac{n\pi x}{\ell} = 0$$

$$\Rightarrow \quad B_n = 0.$$

The first initial condition then reads:

$$\frac{4h}{\ell^2}\, x(\ell-x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} .$$

Hence,

$$A_n = \frac{2}{\ell}\int_0^\ell \frac{4h}{\ell^2}\, x(\ell-x)\sin\frac{n\pi x}{\ell}\,dx .$$

This integral is identical to that worked out in problem (2) (see pages 3 and 4), provided we replace T with h. Hence:

$$A_n = \begin{cases} 0, & n = 2m \\ \dfrac{32h}{n^3\pi^3}, & n = 2m+1, \quad m = 0, 1, \ldots \end{cases}$$

Substituting this into equation (*) on page 8, we have the final answer:

$$u(x,t) = \frac{32h}{\pi^3}\sum_{m=0}^{\infty} \frac{\sin\frac{(2m+1)\pi x}{\ell}}{(2m+1)^3} \cos\frac{(2m+1)\pi c}{\ell}t .$$

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(5) Solve the problem of finding the shape of a rectangular membrane, $-a \le x \le a$, $-b \le y \le b$, under a uniform load q.

The equilibrium shape satisfies Poisson's equation

$$u_{xx} + u_{yy} = -\frac{q}{T}, \qquad (1)$$

with boundary conditions

$$u(-a,y) = u(a,y) = u(x,-b) = u(x,b) = 0 .$$

(a) seek solution in the form:

$$u(x,y) = u_1(x) .$$

Substituting into (1), we have:

$$\frac{d^2u_1}{dx^2} = -\frac{q}{T} .$$

Integrating:
$$\frac{du_1}{dx} = -\frac{q}{T}x + C_1,$$
and again,
$$u_1(x) = -\frac{qx^2}{2T} + C_1 x + C_2.$$
Using the boundary conditions $u_1(-a) = u_1(a) = 0$:
$$-\frac{qa^2}{2T} - C_1 a + C_2 = 0$$
$$-\frac{qa^2}{2T} + C_1 a + C_2 = 0$$
Subtracting the 1st equation from the 2nd gives
$$2C_1 a = 0 \Rightarrow C_1 = 0.$$
Then
$$C_2 = \frac{qa^2}{2T},$$
and
$$\underline{u_1(x) = \frac{q}{2T}(a^2 - x^2)}.$$

(b) Let $u(x,y) = \tilde{u}(x,y) + u_1(x)$.

Substituting into (1):
$$\tilde{u}_{xx} + \tilde{u}_{yy} + \cancel{u_1''(x)} = \cancel{-\frac{q}{T}}.$$
$$\Rightarrow \quad \tilde{u}_{xx} + \tilde{u}_{yy} = 0. \qquad (2)$$

Boundary conditions:
$$\tilde{u}(-a,y) + u_1(-a) = 0 \Rightarrow \tilde{u}(-a,y) = 0. \qquad (3)$$
$$\tilde{u}(a,y) + u_1(a) = 0 \Rightarrow \tilde{u}(a,y) = 0 \qquad (4)$$
$$\tilde{u}(x,-b) + u_1(x) = 0 \Rightarrow \tilde{u}(x,-b) = -u_1(x) \qquad (5)$$
$$\tilde{u}(x,b) + u_1(x) = 0 \Rightarrow \tilde{u}(x,b) = -u_1(x). \qquad (6)$$

(c) Solving (2) by the method of variable separation, we seek solution in the form:

$$\tilde{u}(x,y) = X(x)\,Y(y).$$

$$Y(y)\,X''(x) + X(x)\,Y''(y) = 0 \quad : \text{divide by } XY$$

$$\underbrace{\frac{1}{X(x)}X''(x)}_{\text{depends on } x \text{ only}} + \underbrace{\frac{1}{Y(y)}Y''(y)}_{\text{depends on } y \text{ only}} = 0$$

Must be constant, $-k^2$ const, k^2.

Equation for $X(x)$:

$$X''(x) + k^2 X(x) = 0$$

has the general solution

$$X(x) = A\cos kx + B\sin kx.$$

Using the boundary conditions at $x=-a$ and a: (equations (3) and (4))

$x=-a$: $A\cos ka - B\sin ka = 0$

$x=a$: $A\cos ka + B\sin ka = 0.$

First adding up and then subtracting these equations, we obtain: $A\cos ka = 0$, $B\sin ka = 0$.

Since we are looking for a symmetric (even) solution, we choose $B=0$

and $\cos ka = 0$ gives $ka = (n+\frac{1}{2})\pi$
$n = 0, 1, \ldots$

$$k = \frac{(n+\frac{1}{2})\pi}{a}$$

$$\Rightarrow \quad X(x) = A\cos\frac{(n+\frac{1}{2})\pi x}{a}.$$

The equation for $Y(y)$:

$$Y''(y) - k^2 Y(y) = 0,$$

has the general solution:

$$Y(y) = C_1 e^{ky} + C_2 e^{-ky}.$$

Since we are looking for a <u>symmetric</u> solution ($Y(-y) = Y(y)$), we require $C_2 = C_1$, so that

$$Y(y) = C_1 (e^{ky} + e^{-ky}),$$

$$Y(y) = 2C_1 \cosh ky,$$

or $$Y(y) = 2C_1 \cosh \frac{(n+\frac{1}{2})\pi}{a} y.$$

Combining $X(x)$ and $Y(y)$, we have:

$$\tilde{u}(x,y) = A_n \cosh \frac{(n+\frac{1}{2})\pi y}{a} \cos \frac{(n+\frac{1}{2})\pi x}{a}.$$

where A_n is an arbitrary constant, and $n = 0, 1, 2, \dots$

(d) Using the superposition principle, we construct a more general solution as

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cosh \frac{(n+\frac{1}{2})\pi y}{a} \cos \frac{(n+\frac{1}{2})\pi x}{a}. \quad (*)$$

It must satisfy the boundary conditions (5), (6), and since this function is <u>even</u>, we can apply just one of them, say for $y = b$:

$$\sum_{n=0}^{\infty} \underbrace{A_n \cosh \frac{(n+\frac{1}{2})\pi b}{a}}_{a_n} \cos \frac{(n+\frac{1}{2})\pi x}{a} = -\frac{q}{2T}(a^2 - x^2)$$

This is a Fourier-type series on $-a \le x \le a$, whose coefficients, $a_n = A_n \cosh \frac{(n+\frac{1}{2})\pi b}{a}$, can be found by standard means.

$$a_n = \frac{1}{a}\int_{-a}^{a}\left(-\frac{q}{2T}(a^2-x^2)\right)\cos\frac{(n+\frac12)\pi x}{a}\,dx$$

$$= -\frac{q}{2aT}\int_{-a}^{a}(a^2-x^2)\cos\frac{(n+\frac12)\pi x}{a}\,dx$$

The integrand is even, so we can change $\int_{-a}^{a}$ to $2\int_0^a$

$$= -\frac{q}{aT}\int_0^a (a^2-x^2)\cos\frac{(n+\frac12)\pi x}{a}\,dx$$

$$= -\frac{q}{aT}\left[a^2\int_0^a \cos\frac{(n+\frac12)\pi x}{a}\,dx - \int_0^a x^2\cos\frac{(n+\frac12)\pi x}{a}\,dx\right]$$

$$= -\frac{q}{aT}\left[a^2\frac{a}{(n+\frac12)\pi}\sin\frac{(n+\frac12)\pi x}{a}\Big|_0^a - x^2\frac{a}{(n+\frac12)\pi}\sin\frac{(n+\frac12)\pi x}{a}\Big|_0^a + \frac{2a}{(n+\frac12)\pi}\int_0^a \sin\frac{(n+\frac12)\pi x}{a}\,x\,dx\right]$$

$$= -\frac{q}{aT}\left[\cancel{\frac{a^3(-1)^n}{(n+\frac12)\pi}} - \cancel{\frac{a^3(-1)^n}{(n+\frac12)\pi}} - \frac{2ax}{(n+\frac12)\pi}\cdot\frac{a}{(n+\frac12)\pi}\cos\frac{(n+\frac12)\pi x}{a}\Big|_0^a + \frac{2a^2}{(n+\frac12)^2\pi^2}\int_0^a \cos\frac{(n+\frac12)\pi x}{a}\,dx\right]$$

$\sin(n+\frac12)\pi = (-1)^n$

$\sin 0 = 0$

$\cos(n+\frac12)\pi = 0$

$$= -\frac{q}{aT}\cdot\frac{2a^2}{(n+\frac12)^2\pi^2}\,\frac{a}{(n+\frac12)\pi}\sin\frac{(n+\frac12)\pi x}{a}\Big|_0^a$$

$$= -\frac{2qa^2}{T}\,\frac{(-1)^n}{(n+\frac12)^3\pi^3} = \frac{16qa^2(-1)^{n+1}}{T(2n+1)^3\pi^3}.$$

Writing $\dfrac{(n+\frac12)\pi x}{a} = \dfrac{(2n+1)\pi x}{2a}$, etc., we have

$$A_n = \frac{16qa^2}{T}\,\frac{(-1)^{n+1}}{(2n+1)^3\pi^3}\cdot\frac{1}{\cosh\frac{(2n+1)\pi b}{2a}}$$

and substituting this into (*) on page 12, and combining with $u_1(x)$, we have for $u(x,y) = u_1(x) + \tilde u(x,y)$:

$$u(x,y) = \frac{qa^2}{2T}\left[1 - \frac{x^2}{a^2} + \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cosh\frac{(2n+1)\pi y}{2a}}{(2n+1)^3\cosh\frac{(2n+1)\pi b}{2a}}\cos\frac{(2n+1)\pi x}{2a}\right].$$