AMA3006 Problem sheet 5

Examples

(1) $u_{xx} + u_{yy} = 0$ for $u(x,y)$, $-\infty < x < \infty$, $0 \le y < \infty$.

Boundary condition: $u(x,0) = f(x)$.

(a) $$U(p,y) = \mathcal{F}[u] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} u(x,y)\, dx .$$

1) $$\mathcal{F}[u_{yy}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} \frac{\partial^2 u(x,y)}{\partial y^2}\, dx = \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} u(x,y)\, dx \right)$$

The order of integration over x and differentiation with respect to y can be interchanged.

$$\Longrightarrow \quad \mathcal{F}[u_{yy}] = \frac{\partial^2 U(p,y)}{\partial y^2} .$$

2) $$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} \frac{\partial^2 u}{\partial x^2}\, dx$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{ipx} \frac{\partial u}{\partial x} \Big|_{-\infty}^{+\infty}}_{=0} - \frac{ip}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} \frac{\partial u}{\partial x}\, dx$$

Here we use integration by parts: $\int f g' dx = fg - \int f' g\, dx$

since $\frac{\partial u}{\partial x} \xrightarrow[x \to \pm\infty]{} 0$

$$= \underbrace{-\frac{ip}{\sqrt{2\pi}} e^{ipx} u \Big|_{-\infty}^{+\infty}}_{=0} + \frac{(ip)^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} u(x,y)\, dx = -\frac{p^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ipx} u(x,y)\, dx$$

since $u(x,y) \xrightarrow[x \to \pm\infty]{} 0$

Therefore, $$\mathcal{F}[u_{xx}] = -p^2 U(p,y) .$$

(b) Applying the Fourier transform to the Laplace equation: $\mathcal{F}[u_{xx}]+\mathcal{F}[u_{yy}]=0$, gives:

$$\frac{\partial^2 U}{\partial y^2} - p^2 U = 0 \quad (*)$$

This is an ordinary differential equation with respect to y, p being constant (a parameter). The general solution of $(*)$ is

$$U(p,y) = A(p)\, e^{-py} + B(p)\, e^{py}.$$

The solution we seek should not diverge at $y \to \infty$. Hence, for $p>0$ we must choose e^{-py}, while for $p<0$ we should choose e^{py}. This can be done by writing

$$U(p,y) = A(p)\, e^{-|p|y}. \quad (**)$$

Let us now find $A(p)$. We know from the boundary condition that $u(x,0)=f(x)$. Taking the Fourier transform of this equation, we have:

$$U(p,0) = F(p), \text{ where } F(p) = \mathcal{F}[f].$$

By setting $y=0$ in $(**)$, we obtain:

$$A(p) = F(p),$$

hence

$$U(p,y) = F(p)\, e^{-|p|y}.$$

To find $u(x,y)$ we must perform the inverse transform of $U(p,y)$.

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx}\, U(p,y)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx}\, F(p)\, e^{-|p|y}\, dp$$

$F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ip\xi} f(\xi)\, d\xi$

where we use ξ to avoid confusion when this is substituted.

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{+\infty} e^{-ipx}\, e^{-|p|y}\, dp \int_{-\infty}^{+\infty} e^{ip\xi} f(\xi)\, d\xi$$

Changing the order of integration,

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi)\, d\xi \underbrace{\int_{-\infty}^{+\infty} e^{ip(\xi - x) - |p|y}\, dp}$$

To expand the modulus, $|p|$, we split the integral into two, with $p \geq 0$ and $p \leq 0$.

$$= \int_{-\infty}^{0} e^{ip(\xi - x) + py}\, dp + \int_{0}^{\infty} e^{ip(\xi - x) - py}\, dp$$

$|p| = p$ for $p \geq 0$

$|p| = -p$ for $p \leq 0$

$$= \int_{-\infty}^{0} e^{p(y + i(\xi - x))}\, dp + \int_{0}^{\infty} e^{-p(y - ip(\xi - x))}\, dp$$

$$= \left. \frac{e^{p(y + i(\xi - x))}}{y + i(\xi - x)} \right|_{-\infty}^{0} - \left. \frac{e^{-p(y - ip(\xi - x))}}{y - i(\xi - x)} \right|_{0}^{\infty}$$

only upper limit contributes here — only lower limit contributes here

$$= \frac{1}{y + i(\xi - x)} + \frac{1}{y - i(\xi - x)} = \frac{y - i(\xi - x) + y + i(\xi - x)}{(y + i(\xi - x))(y - i(\xi - x))}$$

$$= \frac{2y}{y^2 + (\xi - x)^2}$$

Substituting this into the integral over ξ:

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)\, d\xi}{y^2 + (\xi - x)^2} \,.$$

(c) Let us calculate $u(x,y)$ for the special case $f(x) = T$ for $-a \leq x \leq a$, and 0 otherwise.
Substituting this in the last equation,

$$u(x,y) = \frac{y}{\pi}\int_{-a}^{a} \frac{T\,d\xi}{y^2+(\xi-x)^2}$$

$$= \frac{yT}{\pi}\int_{-a}^{a} \frac{d\xi}{y^2+(\xi-x)^2}$$

$$= \frac{yT}{\pi}\,\frac{1}{y^2}\int_{-a}^{a} \frac{d\xi}{1+\left(\frac{\xi-x}{y}\right)^2}$$

We know a table integral $\int \frac{dx}{1+x^2} = \arctan x$

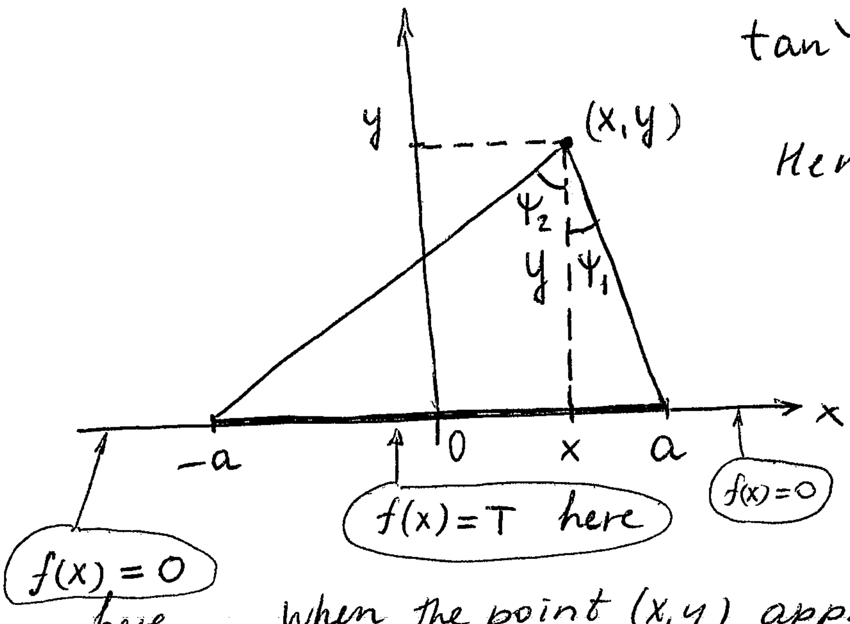
$$= \frac{yT}{\pi}\,\frac{y}{y^2}\int_{-a}^{a} \frac{d\left(\frac{\xi-x}{y}\right)}{1+\left(\frac{\xi-x}{y}\right)^2}$$

$$= \frac{T}{\pi}\left[\arctan\frac{\xi-x}{y}\right]_{-a}^{a} = \frac{T}{\pi}\left[\arctan\frac{a-x}{y} - \arctan\frac{-a-x}{y}\right]$$

arctan is an odd function, hence we can write:

$$u(x,y) = \frac{T}{\pi}\left[\arctan\frac{a-x}{y} + \arctan\frac{a+x}{y}\right]$$

This result has an interesting geometric interpretation.



$\tan\Psi_1 = \frac{a-x}{y}$, $\tan\Psi_2 = \frac{a+x}{y}$

Hence

$$u(x,y) = \frac{T}{\pi}\left[\Psi_1 + \Psi_2\right],$$

and $\Psi_1 + \Psi_2$ is the angle subtended by the $(-a, a)$ segment of the x axis.

When the point (x,y) approaches the x axis within $-a < x < a$, $\Psi_1 + \Psi_2 \to \pi$ and $u(x,y) \to T$, as required.

(2) Heat equation: $u_t - u_{xx} = 0$ for $u(x,t)$, $0 \le x < \infty$, with initial condition $u(x,0) = 0$ and boundary condition $u_x(0,t) = -\sigma$.

[Recall: $j = -\varkappa \frac{\partial u}{\partial x}$ is the heat flux, so that $\left.\frac{\partial u}{\partial x}\right|_{x=0} = -\sigma$ means $j = +\varkappa\sigma$ - constant heat flux into the rod.]

Using the cosine Fourier transform:

$$U_c(p,t) = \mathcal{F}_c[u] = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos px \, dx .$$

Fourier transform of u_t:

$$\mathcal{F}[u_t] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos px \, dx = \frac{\partial}{\partial t}\left(\sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos px \, dx \right)$$

$$\Rightarrow \quad \mathcal{F}[u_t] = \frac{\partial U_c(p,t)}{\partial t} .$$

Fourier transform of u_{xx}:

$$\mathcal{F}[u_{xx}] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos px \, dx \quad \Big\} \text{ Integrating by parts}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\partial u}{\partial x} \cos px \Big|_0^\infty + \sqrt{\frac{2}{\pi}} p \int_0^\infty \frac{\partial u}{\partial x} \sin px \, dx$$

$$= -\sqrt{\frac{2}{\pi}} u_x(0,t) + \underbrace{\sqrt{\frac{2}{\pi}} p \, u(x,t) \sin px \Big|_0^\infty}_{\text{This is zero both at } x=\infty \text{ and } x=0} - \sqrt{\frac{2}{\pi}} p^2 \int_0^\infty u(x,t) \cos px \, dx$$

only the lower limit $x = 0$ contributes here; we assume $\frac{\partial u}{\partial x} \underset{x\to\infty}{\longrightarrow} 0$ and $u \underset{x\to\infty}{\longrightarrow} 0$

Therefore: $$\mathcal{F}[u_{xx}] = -\sqrt{\frac{2}{\pi}} u_x(0,t) - p^2 U_c(p,t) .$$

Let us apply the cosine Fourier transform to the heat equation: $\mathcal{F}[u_t] - \mathcal{F}[u_{xx}] = 0$,

which gives: $$\frac{\partial U_c(p,t)}{\partial t} + \sqrt{\frac{2}{\pi}}\, u_x(0,t) + p^2 U_c(p,t) = 0,$$

or $$\frac{\partial U_c(p,t)}{\partial t} - \sqrt{\frac{2}{\pi}}\,\sigma + p^2 U_c(p,t) = 0.$$

$$\frac{\partial U_c}{\partial t} + p^2 U_c = \sqrt{\frac{2}{\pi}}\,\sigma . \qquad (*)$$

This is a first-order linear ordinary differential equation. The homogeneous equation is

$$\frac{\partial U_c}{\partial t} + p^2 U_c = 0,$$

has the solution $U_c = C(p)\, e^{-p^2 t}$.

To solve the inhomogeneous equation, replace the "constant" $C(p)$ with a function of t, $C(p,t)$.

$$U_c = C(p,t)\, e^{-p^2 t}.$$

Substituting into $(*)$:

$$\frac{\partial C}{\partial t} e^{-p^2 t} + \cancel{C(p,t)(-p^2)e^{-p^2 t}} + \cancel{p^2 C(p,t) e^{-p^2 t}} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\frac{\partial C}{\partial t} = \sqrt{\frac{2}{\pi}}\,\sigma\, e^{p^2 t}$$

Integrating over t:

$$C(p,t) = \sqrt{\frac{2}{\pi}}\,\frac{\sigma}{p^2}\, e^{p^2 t} + C_1(p)$$

arbitrary constant (function of p only)

$$\Rightarrow \quad U_c(p,t) = \left(\sqrt{\frac{2}{\pi}}\,\frac{\sigma}{p^2}\, e^{p^2 t} + C_1(p)\right) e^{-p^2 t}$$

$$U_c(p,t) = C_1(p)\, e^{-p^2 t} + \sqrt{\frac{2}{\pi}}\,\frac{\sigma}{p^2}. \qquad (**)$$

Using the initial condition, $u(x,0)=0$, we obtain: $$U_c(p,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \cos px \, dx = 0.$$

Setting $t=0$ in equation $(**)$ (page 6):

$$C_1(p) + \sqrt{\frac{2}{\pi}} \frac{\sigma}{p^2} = 0 \Rightarrow C_1(p) = -\sqrt{\frac{2}{\pi}} \frac{\sigma}{p^2}.$$

Hence:

$$U_c(p,t) = \sqrt{\frac{2}{\pi}} \, \sigma \, \frac{1-e^{-p^2 t}}{p^2}.$$

From this, $u(x,t)$ is obtained by performing the inverse cosine Fourier transform:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_c(p,t) \cos px \, dp,$$

i.e.,

$$u(x,t) = \frac{2\sigma}{\pi} \int_0^\infty \frac{1-e^{-p^2 t}}{p^2} \cos px \, dp.$$

This integral can, in fact, be expressed in terms of the error function, $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$,

as shown in Problem 355 of the book by N.N. Lebedev, I.P. Skal'skaya and Ya.S. Uflyand, Problems in Mathematical Physics (Pergamon Press, Oxford, 1966):

$$u(x,t) = \sigma \left\{ \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-\frac{x^2}{4t}} - x \left[1 - \mathrm{erf}\left(\frac{x}{2\sqrt{t}} \right) \right] \right\}.$$

In fact, this is not too difficult to demonstrate (see next page if you are curious!).

Let's differentiate $u(x,t)$ with respect to t:

$$\frac{\partial u(x,t)}{\partial t} = \frac{2\sigma}{\pi}\int_0^\infty e^{-p^2 t}\cos px\,dp \quad :$$

this is an <u>even</u> function, change $\int_0^\infty$ to $\frac{1}{2}\int_{-\infty}^{+\infty}$.

$$= \frac{2\sigma}{\pi}\frac{1}{2}\frac{1}{2}\int_{-\infty}^{\infty} e^{-p^2 t}\left(e^{ipx}+e^{-ipx}\right)dp$$

$$= \frac{\sigma}{2\pi}\left[\int_{-\infty}^{\infty} e^{-tp^2+ipx}dp + \int_{-\infty}^{\infty} e^{-tp^2-ipx}dp\right]$$

Using $\int_{-\infty}^{\infty} e^{-\alpha x^2-\beta x}dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}}$

$$= \frac{\sigma}{2\pi}\left(\sqrt{\frac{\pi}{t}}\, e^{\frac{(ix)^2}{4t}} + \sqrt{\frac{\pi}{t}}\, e^{\frac{(-ix)^2}{4t}}\right) = \frac{\sigma}{\sqrt{\pi}}\frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4t}}.$$

To obtain $u(x,t)$ we need to <u>integrate</u> this with respect to t from 0 to t ($u(x,0)=0$).

$$\Rightarrow \quad u(x,t) = \frac{\sigma}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4t}}\,dt$$

Let us introduce the new variable $s^2 = \frac{x^2}{4t}$, $s = \frac{x}{2\sqrt{t}}$.

$ds = -\frac{x}{2\cdot 2\, t^{3/2}}\,dt$ or $t = \frac{x^2}{4s^2}$, $dt = -\frac{x^2}{2s^3}\,ds$,

so that: $$u(x,t) = \frac{\sigma}{\sqrt{\pi}}\int_{x/2\sqrt{t}} \frac{2s}{x}\, e^{-s^2}\left(-\frac{x^2}{2s^3}\right)ds$$

$$= \frac{\sigma}{\sqrt{\pi}}\frac{x^2}{x}\int_{x/2\sqrt{t}}^{\infty} e^{-s^2}\frac{ds}{s^2} = \frac{\sigma x}{\sqrt{\pi}}\int_{x/2\sqrt{t}}^{\infty} e^{-s^2}\, d\left(-\frac{1}{s}\right)$$

Integrating by parts

$$= \frac{\sigma x}{\sqrt{\pi}}\left[e^{-s^2}\left(-\frac{1}{s}\right)\Big|_{x/2\sqrt{t}}^{\infty} + \int_{x/2\sqrt{t}}^{\infty}\frac{1}{s}\, e^{-s^2}(-2s)\,ds\right]$$

$\int_0^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}$

$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}ds = 1$

$$= \frac{\sigma x}{\sqrt{\pi}}\left[+e^{-\frac{x^2}{4t}}\,\frac{2\sqrt{t}}{x} - \frac{2}{\sqrt{\pi}}\sqrt{\pi}\left(\int_0^\infty - \int_0^{x/2\sqrt{t}}\right)e^{-s^2}ds\right]$$

$$= \sigma\left[\frac{2\sqrt{t}}{\sqrt{\pi}}\, e^{-\frac{x^2}{4t}} - x\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)\right],$$ <u>as in the book quoted.</u>