AMA3006 Problem sheet 5

Homework problems

SOLUTIONS

(1) For $u(x,y)$, $0 \le x < \infty$, the cosine Fourier transform with respect to x is:

$$U_c(p,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,y) \cos px \, dx .$$

$$\mathcal{F}_c[u_{yy}] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial y^2} \cos px \, dx$$

$$= \frac{\partial^2}{\partial y^2} \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,y) \cos px \, dx$$

Integration over x and differentiation with respect to y are interchangeable

$$\Rightarrow \quad \mathcal{F}_c[u_{yy}] = \frac{\partial^2 U_c(p,y)}{\partial y^2} .$$

$$\mathcal{F}_c[u_{xx}] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos px \, dx$$

Integrating by parts

$$= \sqrt{\frac{2}{\pi}} \frac{\partial u}{\partial x} \cos px \Big|_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty (-p \sin px) \frac{\partial u}{\partial x} \, dx$$

The contribution of the upper limit vanishes since $\frac{\partial u}{\partial x} \xrightarrow[x\to\infty]{} 0$

$$= -\sqrt{\frac{2}{\pi}} u_x(0,y) + \underbrace{\sqrt{\frac{2}{\pi}}\, p u \sin px \Big|_0^\infty}_{\substack{= 0 \\ (u \xrightarrow[x\to\infty]{} 0)}} - p^2 \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty \cos px \, u \, dx}_{U_c(p,y)}$$

Therefore:

$$\mathcal{F}_c[u_{xx}] = -\sqrt{\frac{2}{\pi}} u_x(0,y) - p^2 U_c(p,y) .$$

② $u_{xx} + u_{yy} = 0, \quad 0 \le x < \infty, \quad 0 \le y < \infty,$

boundary conditions: $\quad u_x(0,y) = 0$

$$u(x,0) = \begin{cases} T, & 0 \le x \le a \\ 0, & a < x < \infty \end{cases}$$

(a) Applying the cosine Fourier transform to $u_{xx} + u_{yy} = 0$, we have:

$$-\sqrt{\frac{2}{\pi}}\, \underbrace{u_x(0,y)}_{=0} - p^2 U_c + \frac{\partial^2 U_c}{\partial y^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 U_c}{\partial y^2} - p^2 U_c = 0.$$

The general solution of this equation is:

$$U_c(p,y) = A(p)\, e^{-py} + B(p)\, e^{py}.$$

We require that the solution does not diverge at $y \to \infty$, hence, we set $B(p) = 0$.

$$\Rightarrow \quad U_c(p,y) = A(p)\, e^{-py}.$$

Applying the boundary condition at $y = 0$:

$$U_c(p,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a T \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, T \left. \frac{\sin px}{p} \right|_0^a = T \sqrt{\frac{2}{\pi}} \frac{\sin pa}{p}.$$

Hence, $\quad A(p) = T \sqrt{\frac{2}{\pi}} \frac{\sin pa}{p}$, and

$$U_c(p,y) = T \sqrt{\frac{2}{\pi}} \frac{\sin pa}{p} e^{-py}.$$

(b) To find $u(x,y)$ we perform the inverse cosine Fourier transform,

$$u(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_c(p,y) \cos px \, dp$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty T\sqrt{\frac{2}{\pi}} \frac{\sin pa}{p} e^{-py} \cos px \, dp$$

$$= \frac{2T}{\pi} \int_0^\infty \frac{\sin pa \cos px}{p} e^{-py} dp$$

$$= \frac{T}{\pi} \int_0^\infty \frac{\sin(a+x)p + \sin(a-x)p}{p} e^{-py} dp$$

$\sin pa \cos px = \frac{1}{2}[\sin(a+x)p + \sin(a-x)p]$

$$= \frac{T}{\pi}\left[\arctan\frac{a+x}{y} + \arctan\frac{a-x}{y}\right].$$

Using $\int_0^\infty \frac{\sin \alpha p}{p} e^{-\beta p} dp = \arctan\frac{\alpha}{\beta}$

Note that this answer is identical to that of Example 1(c).

The solution obtained is even, $u(-x,y) = u(x,y)$, and solving for $-\infty < x < \infty$ with $u(x,0) = T$ for $-a \le x \le a$ in Example 1, is equivalent to the problem solved in this question. The "evenness" of the solution is guaranteed by the condition $u_x(0,y) = 0$ and even character of the cosine function in the cosine Fourier transform.

③ Heat equation: $u_t - u_{xx} = 0$, $0 \le x < \infty$

(here $K = 1$), initial condition $u(x,0) = 0$,

boundary condition $u(0,t) = T$.

(a) The sine Fourier transform:

$$U_s(p,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin px \, dx .$$

$$\mathcal{F}_s[u_t] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin px \, dx$$

$$= \frac{\partial}{\partial t} \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin px \, dx$$

Integration over x and differentiation with respect to t are interchangeable.

$$\Rightarrow \quad \mathcal{F}_s[u_t] = \frac{\partial U_s(p,t)}{\partial t}$$

$$\mathcal{F}_s[u_{xx}] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

Integrating by parts

$$= \underbrace{\sqrt{\frac{2}{\pi}} \frac{\partial u}{\partial x} \sin px \Big|_0^\infty}_{=0} - \sqrt{\frac{2}{\pi}} \int_0^\infty p \cos px \frac{\partial u}{\partial x} \, dx$$

since $\frac{\partial u}{\partial x} \xrightarrow[x \to \infty]{} 0$

Contribution of upper limit is zero, as $u \xrightarrow[x \to \infty]{} 0$.

$$= -\sqrt{\frac{2}{\pi}} \, p \cos px \, u \Big|_0^\infty - \sqrt{\frac{2}{\pi}} \, p^2 \int_0^\infty \sin px \, u(x,t) \, dx$$

$$\Rightarrow \quad \underline{\mathcal{F}_s[u_{xx}] = \sqrt{\frac{2}{\pi}} \, u(0,t) p - p^2 U_s(p,t) .}$$

(b) Applying the sine Fourier transform to the heat equation:

$$\frac{\partial U_s}{\partial t} - \sqrt{\frac{2}{\pi}} \, u(0,t) p + p^2 U_s = 0$$

Using the boundary condition:

$$\frac{\partial U_s}{\partial t} + p^2 U_s - \sqrt{\frac{2}{\pi}} \, T p = 0 .$$

(c) Let us re-write the above equation:

$$\frac{\partial U_s}{\partial t} + p^2 U_s = \sqrt{\frac{2}{\pi}}\, Tp . \qquad (*)$$

This is a linear 1st-order inhomogeneous differential equation in variable t (p being a parameter). Solving the homogeneous equation first,

$$\frac{\partial U}{\partial t} + p^2 U_s = 0 ,$$

we obtain: $\quad U_s = A(p)\, e^{-p^2 t}$,

where $A(p)$ is an arbitrary function of p (constant with respect to t). To solve the inhomogeneous equation, replace $A(p)$ with a function of t (and p): $\quad A(p) \longrightarrow A(p,t)$

$$U_s = A(p,t)\, e^{-p^2 t}, \qquad (1)$$

and substitute into $(*)$:

$$\frac{\partial A}{\partial t} e^{-p^2 t} + \cancel{A(-p^2) e^{-p^2 t}} + \cancel{p^2 A e^{-p^2 t}} = \sqrt{\frac{2}{\pi}}\, Tp$$

$$\frac{\partial A}{\partial t} = \sqrt{\frac{2}{\pi}}\, Tp\, e^{p^2 t}$$

$$A = \int \sqrt{\frac{2}{\pi}}\, Tp\, e^{p^2 t}\, dt$$

$$A(p,t) = \sqrt{\frac{2}{\pi}}\, T \frac{1}{p}\, e^{p^2 t} + C(p) \qquad (2)$$

arbitrary "constant" (function of p).

$C(p)$ can be determined from the initial condition: $\quad u(x,0) = 0$.

$$U_s(p,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \sin px\, dx = 0$$

Setting $t = 0$ in (1) and using $A(p,t)$ from (2),

$$\sqrt{\frac{2}{\pi}}\,\frac{T}{p} + C(p) = 0$$

$$\Rightarrow \quad C(p) = -\sqrt{\frac{2}{\pi}}\,\frac{T}{p}.$$

Substituting this into (2) and then into (1), we have:

$$A(p,t) = \sqrt{\frac{2}{\pi}}\,\frac{T}{p}\,e^{p^2t} - \sqrt{\frac{2}{\pi}}\,\frac{T}{p}$$

$$= \sqrt{\frac{2}{\pi}}\,T\,\frac{e^{p^2t}-1}{p}.$$

$$U_s(p,t) = \sqrt{\frac{2}{\pi}}\,T\,\frac{e^{p^2t}-1}{p}\,e^{-p^2t}$$

$$U_s(p,t) = \sqrt{\frac{2}{\pi}}\,T\,\frac{1-e^{-p^2t}}{p}. \qquad (3)$$

The function $u(x,t)$ is found by performing the inverse sine Fourier transform:

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty U_s(p,t)\sin px\,dp,$$

and substituting (3), we obtain:

$$u(x,t) = \frac{2T}{\pi}\int_0^\infty \frac{1-e^{-p^2t}}{p}\sin px\,dp.$$

The remaining integral can in fact be expressed in terms of the error function $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$, see end of next problem. This can also be done using two table integrals: $\int_0^\infty \frac{\sin as}{s}ds = \frac{\pi}{2}$ $(a>0)$ and $\int_0^\infty e^{-\beta s^2}\frac{\sin as}{s}ds = \frac{\pi}{2}\operatorname{erf}\left(\frac{a}{2\sqrt{\beta}}\right)$, which gives:

$$u(x,t) = T\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right].$$

(4) Heat equation : $u_t - u_{xx} = 0$ $(K=1)$

with initial condition and boundary condition

$$u(x,0)=0, \qquad u(0,t) = f(t).$$

Applying the sine Fourier transform to the heat equation, we have:

$$\frac{\partial U_s}{\partial t} - \sqrt{\frac{2}{\pi}}\, u(0,t)p + p^2 U_s = 0.$$

(see question 3, part (a)).

$$\frac{\partial U_s}{\partial t} + p^2 U_s = \sqrt{\frac{2}{\pi}}\, f(t)\, p. \qquad (1)$$

This is a 1st-order linear inhomogeneous equation. Solving the homogeneous equation first,

$$\frac{\partial U_s}{\partial t} + p^2 U_s = 0$$

$$U_s = A(p)\, e^{-p^2 t}.$$

To solve the inhomogeneous equation, replace $A(p)$ with a function of t (and p):

$$U_s(p,t) = A(p,t)\, e^{-p^2 t},$$

and substitute into (1):

$$\frac{\partial A}{\partial t} e^{-p^2 t} + \cancel{A(-p^2) e^{-p^2 t}} + \cancel{p^2 A e^{-p^2 t}} = \sqrt{\frac{2}{\pi}}\, f(t)\, p\, e^{p^2 t}$$

$$\frac{\partial A}{\partial t} = \sqrt{\frac{2}{\pi}}\, p\, f(t)\, e^{p^2 t} \qquad (2')$$

$$A(p,t) = \int \sqrt{\frac{2}{\pi}}\, p\, f(t)\, e^{p^2 t}\, dt + \text{const} \qquad (2)$$

The initial condition, $u(x,0)=0$, requires $U_s(p,0)=0$, which in turn means $A(p,0) = 0$.

This can be fulfilled by replacing the indefinite integral in (2) by a definite integral with the limits 0 and t, and setting const $= 0$:

$$A(p,t) = \sqrt{\tfrac{2}{\pi}}\, p \int_0^t e^{p^2\tau} f(\tau)\, d\tau . \qquad (3)$$

[Here we replaced the integration variable by τ, to avoid confusion with the upper limit t. Obviously, (3) satisfies (2') and $A(p,0)=0$, as required.]

Hence: $$U_s(p,t) = \sqrt{\tfrac{2}{\pi}}\, p\, e^{-p^2 t} \int_0^t e^{p^2\tau} f(\tau)\, d\tau .$$

To find $u(x,t)$, substitute $U_s(p,t)$ in the inverse sine Fourier transform equation,

$$u(x,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty U_s(p,t) \sin px \, dp$$

$$u(x,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sqrt{\tfrac{2}{\pi}}\, p\, e^{-p^2 t} \int_0^t e^{p^2\tau} f(\tau)\, d\tau \, \sin px \, dp$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^t e^{-p^2(t-\tau)}\, p \sin px \, f(\tau)\, d\tau\, dp$$

Changing the order of integration:

$$u(x,t) = \frac{2}{\pi} \int_0^t f(\tau)\, d\tau \int_0^\infty e^{-p^2(t-\tau)}\, p \sin px \, dp$$

and using $$\int_0^\infty e^{-\alpha p^2}\, p \sin \beta p \, dp = \sqrt{\pi}\, e^{-\frac{\beta^2}{4\alpha}} \frac{\beta}{4\alpha^{3/2}} ,$$

with $\alpha = t-\tau$, $\beta = x$, to integrate over p, we obtain:

$$u(x,t) = \frac{2}{\pi}\int_0^t f(\tau)\,d\tau \;\sqrt{\pi}\; e^{-\frac{x^2}{4(t-\tau)}} \frac{x}{4(t-\tau)^{3/2}}$$

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_0^t f(\tau)\, e^{-\frac{x^2}{4(t-\tau)}} \frac{x}{4(t-\tau)^{3/2}}\,d\tau .$$

Let us introduce new variable, $s = \frac{x}{2\sqrt{t-\tau}}$

$$s^2 = \frac{x^2}{4(t-\tau)}, \qquad \sqrt{t-\tau} = \frac{x}{2s}$$

$$t-\tau = \frac{x^2}{4s^2}, \qquad \tau = t - \frac{x^2}{4s^2}$$

$$d\tau = \frac{x^2}{2s^3}\,ds$$

Limits: $\tau = 0 \qquad s = \frac{x}{2\sqrt{t}}$

$\tau = t \qquad s \to \infty$

Performing all the changes in the above integral:

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} f\left(t - \frac{x^2}{4s^2}\right) e^{-s^2} \frac{x\,8s^3}{4x^3}\,\frac{x^2}{2s^3}\,ds$$

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} f\left(t - \frac{x^2}{4s^2}\right) e^{-s^2}\,ds .$$

For $f(t) = T$ (as in problem 3), this expression gives:

$$u(x,t) = \frac{2T}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-s^2}\,ds = \frac{2T}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-s^2}\,ds - \int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2}\,ds\right]$$

$$= T\left[\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-s^2}\,ds - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2}\,ds\right]$$

$$= T\left[\underbrace{\mathrm{erf}(\infty)}_{=1} - \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] = T\left[1 - \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]$$

This is the final answer to problem 3.