Partial Differential Equations AMA3006 **Problem sheet 6**

Laplace transform and its application to ODE and PDE.

The Laplace transform of a piecewise smooth function $f(t)$ ($f(t) = 0$ for $t < 0$) is

$$F(p) \equiv \mathcal{L}[f] = \int_0^\infty f(t)e^{-pt}dt. \tag{1}$$

Its inverse is an integral in the complex p plane along the line parallel to the imaginary axis,

$$f(t) = \mathcal{L}^{-1}[F] = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(p)e^{pt}dp, \tag{2}$$

where the integration path is chosen so that $F(p)$ is regular for $\mathrm{Re}\, p > \sigma$.
In many cases there is no need to perform the inverse, as one can determine the original $f(t)$ by recognising its $F(p)$. In particular, this can be done with the help of the *convolution theorem*:

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = F(p)G(p), \tag{3}$$

where $G(p) = \mathcal{L}[g]$, and the quantity in brackets is the *convolution* of functions f and g.

Examples

1. Show that:

   (a) For a function $y(t)$, $\mathcal{L}[y''] = p^2Y(p) - py(0) - y'(0)$,

   (b) $\mathcal{L}[e^{\alpha t}] = \dfrac{1}{p-\alpha}$,

   (c) $\mathcal{L}[te^{\alpha t}] = \dfrac{1}{(p-\alpha)^2}$,

   (d) $\mathcal{L}[\cos\omega t] = \dfrac{p}{p^2+\omega^2}$.

   (e) $\mathcal{L}[\sin\omega t] = \dfrac{\omega}{p^2+\omega^2}$,

   (f) $\mathcal{L}[\theta(t-s)] = \dfrac{e^{-ps}}{p}$ $(s \geq 0)$, where $\theta(t)$ is the Heaviside step function:

   $$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

2. Use the Laplace transform to solve for $y(t)$:

   (a) $y'' + y = \sin 2t, \quad y(0) = 0,\ y'(0) = 0,$

   (b) $y'' + y = \sin t, \quad y(0) = 0,\ y'(0) = 0.$

3. Consider the wave equation for $u(x,t)$ for a semi-infinite string, $0 \leq x < \infty$,

   $$u_{tt} - c^2u_{xx} = 0,$$

   with the initial and boundary conditions $u(x,0) = u_t(x,0) = 0$, $u_t(0,t) = g(t)$.

   Using the Laplace transform with respect to t, show that

   $$u(x,t) = \begin{cases} \int_0^{t-x/c} g(\tau)d\tau, & x \leq ct, \\ 0, & x > ct. \end{cases}$$

Homework problems

1. By using the definition (1), prove the shift theorems for $f(t)$ and $F(p) = \mathcal{L}[f]$:
   (a) $\mathcal{L}[e^{\alpha t} f(t)] = F(p - \alpha)$,
   (b) $\mathcal{L}[f(t - a)] = e^{-pa} F(p)$.

2. Show that for a function $y(t)$, $\mathcal{L}[y'] = pY(p) - y(0)$.

3. Use Laplace transform to find $y(t)$ that satisfies

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \; y'(0) = -1.$$

   Hint: present $Y(p)$ in the form $\frac{A}{p-2} + \frac{B}{p-1}$ with suitable A and B.

   Answer: $y(t) = -2e^{2t} + 3e^t$.

4. Use the Laplace transform to solve $y'' + 2y' = e^{-t}$, subject to $y(0) = y'(0) = 0$.
   Hint: Use partial fractions to show that $Y(p) = \frac{1}{2p} - \frac{1}{p+1} + \frac{1}{2(p+2)}$.

5. (a) Show that the Laplace transform of the solution $y(t)$ of the equation

$$y'' + \omega^2 y = f(t), \tag{4}$$

   is given by

$$Y(p) = \frac{F(p) + y'(0) + py(0)}{p^2 + \omega^2}.$$

   (b) Hence, show that a particular solution of (4) for which $y(0) = y'(0) = 0$, is

$$y(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega(t - \tau)\, d\tau.$$

   Hint: Use the convolution theorem.

6. Using the Laplace transform, solve the coupled equations for $y(t)$ and $z(t)$,

$$y' = 4y - 2z, \quad z' = 5y + 2z, \quad \text{subject to} \quad y(0) = 2, \; z(0) = -2.$$

   Hints: Show that $Y(p) = \frac{2p}{p^2-6p+18}$, $Z(p) = \frac{-2p+18}{p^2-6p+18}$, and re-write these as

$$Y(p) = \frac{2(p-3)}{(p-3)^2+9} + \frac{6}{(p-3)^2+9}, \quad Z(p) = \frac{-2(p-3)}{(p-3)^2+9} + \frac{12}{(p-3)^2+9}.$$

   Then use examples 1(d) and 1(e) and the first shift theorem to find $y(t)$ and $z(t)$.

7. Consider the wave equation for a semi-infinite string, $0 \le x < \infty$,

$$u_{tt} - c^2 u_{xx} = 0, \tag{5}$$

   with initial conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$, and boundary condition $u(0, t) = f(t)$. Using the Laplace transform with respect to t, $U(x, p) = \mathcal{L}[u]$, and applying it to (5), show that

$$U(x, p) = F(p) e^{-px/c}, \quad \text{where} \quad F(p) = \mathcal{L}[f].$$

   Hence, by using the second shift theorem, prove that

$$u(x, t) = \begin{cases} f(t - x/c), & x \le ct, \\ 0, & x > ct. \end{cases}$$

   Comment: The above answer shows that the displacement at point x lags behind that at the origin by x/c, the time it takes the wave to reach point x. The points at $x > ct$ remain stationary, as they have not been reached by the wave yet.