AMA3006 Problem sheet 7

Homework problems

SOLUTIONS

① $\frac{1}{\sqrt{2\pi}}$, $\frac{\cos nx}{\sqrt{\pi}}$, $\frac{\sin nx}{\sqrt{\pi}}$ $(n = 1, 2, \ldots)$.

Let us first check that the norms of these functions are equal to unity.

1) $$\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = \frac{1}{2\pi}\int_0^{2\pi} dx = \frac{1}{2\pi}\, x\Big|_0^{2\pi} = \frac{2\pi}{2\pi} = 1.$$

2) $$\int_0^{2\pi} \left(\frac{\cos nx}{\sqrt{\pi}}\right)^2 dx = \frac{1}{\pi}\int_0^{2\pi} \cos^2 nx\, dx$$

[Using $\cos 2\theta = 2\cos^2\theta - 1 \iff \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$]

$$= \frac{1}{2\pi}\int_0^{2\pi} (1+\cos 2nx)\,dx = \frac{1}{2\pi}\left[\int_0^{2\pi} dx + \int_0^{2\pi}\cos 2nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[2\pi + \frac{1}{2n}\sin 2nx\Big|_0^{2\pi}\right] = \frac{1}{2\pi}\left[2\pi + 0\right] = 1.$$

$(\sin 4n\pi = 0)$.

3) $$\int_0^{2\pi} \left(\frac{\sin nx}{\sqrt{\pi}}\right)^2 dx = \frac{1}{\pi}\int_0^{2\pi} \sin^2 nx\, dx$$

[Using $\cos 2\theta = 1 - 2\sin^2\theta \iff \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$]

$$= \frac{1}{2\pi}\int_0^{2\pi} (1-\cos 2nx)\,d\theta = \frac{1}{2\pi}\left[2\pi - \int_0^{2\pi}\cos 2nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[2\pi - 0\right] = 1.$$

Checking the orthogonality of different functions:

1. $$\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos nx}{\sqrt{\pi}} dx = \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} \cos nx \, dx = \frac{1}{\sqrt{2}\,\pi} \frac{1}{n} \sin nx \Big/_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\,\pi n} (\sin 2n\pi - \sin 0) = \underline{\underline{0}}.$$

2. $$\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin nx}{\sqrt{\pi}} dx = \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} \sin nx \, dx = -\frac{1}{\sqrt{2}\,\pi n} \cos nx \Big/_0^{2\pi}$$

$$= -\frac{1}{\sqrt{2}\,\pi n} (\underbrace{\cos 2n\pi}_{=1} - \underbrace{\cos 0}_{=1}) = \underline{\underline{0}}.$$

3. $$\int_0^{2\pi} \frac{\cos nx}{\sqrt{\pi}} \frac{\cos mx}{\sqrt{\pi}} dx = \frac{1}{\pi} \int_0^{2\pi} \cos nx \cos mx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [\cos (n+m)x + \cos (n-m)x] dx$$

$$= \frac{1}{2\pi} \left[\frac{1}{n+m} \sin (n+m)x + \frac{1}{n-m} \sin (n-m)x \right]_0^{2\pi} = \underline{\underline{0}}.$$

$(n \neq m)$

4. $$\int_0^{2\pi} \frac{\sin nx}{\sqrt{\pi}} \frac{\sin mx}{\sqrt{\pi}} dx = \frac{1}{\pi} \int_0^{2\pi} \sin nx \sin mx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [\cos (n-m)x - \cos (n+m)x] dx$$

$$= \frac{1}{2\pi} \left[\frac{1}{n-m} \sin (n-m)x - \frac{1}{n+m} \sin (n+m)x \right]_0^{2\pi} = \underline{\underline{0}}.$$

5. $$\int_0^{2\pi} \frac{\sin nx}{\sqrt{\pi}} \frac{\cos mx}{\sqrt{\pi}} dx = \frac{1}{\pi} \int_0^{2\pi} \sin nx \cos mx \, dx$$

For $n = m$
$\sin (n-m)x = 0$

$$= \frac{1}{2\pi} \int_0^{2\pi} [\sin (n+m)x + \sin (n-m)x] dx$$

$n \neq m$

$$= \frac{1}{2\pi} \left[-\frac{1}{n+m} \cos (n+m)x - \frac{1}{n-m} \cos (n-m)x \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-\frac{1}{n+m} (1-1) - \frac{1}{n-m} (1-1) \right] = \underline{\underline{0}}.$$

②

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

(a) From Example 1 : $\varphi_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)$.

$n = 1$:

$$P_1(x) = \frac{1}{2} \frac{d(x^2-1)}{dx} = \frac{1}{2} 2x = x$$

$$\underline{\varphi_1(x) = \sqrt{\frac{3}{2}}\, x}\ .$$

$n = 2$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2 (x^2-1)^2}{dx^2}$$

$$= \frac{1}{4 \cdot 2} \frac{d}{dx}\left(2(x^2-1)2x \right)$$

$$= \frac{1}{8} 4(2x \cdot x + x^2 - 1)$$

$$= \frac{1}{2}(3x^2 - 1)$$

$$\underline{\varphi_2(x) = \sqrt{\frac{5}{2}}\ \frac{3x^2-1}{2}}\ .$$

$n = 3$

$$P_3(x) = \frac{1}{2^3 3!} \left((x^2-1)^3\right)'''$$

$$= \frac{1}{8 \cdot 6} \left(3(x^2-1)^2 2x\right)''$$

$$= \frac{1}{8} \left(2(x^2-1)\, 2x\ x + (x^2-1)^2\right)'$$

$$= \frac{1}{8}\left(4(x^2-1)x^2 + (x^2-1)^2 \right)'$$

$$= \frac{1}{8}\left(4 \cdot 2x\, x^2 + 4(x^2-1)2x + 2(x^2-1)2x \right)$$

$$= \frac{1}{2}(2x^3 + 2x^3 - 2x + x^3 - x)$$

$$= \frac{5x^3 - 3x}{2}$$

$$\Rightarrow \quad \underline{\varphi_3(x) = \sqrt{\frac{7}{2}} \cdot \frac{5x^3-3x}{2}}\ .$$

(b) $$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} x^m \left[(x^2-1)^n\right]^{(n)} dx$$

Considering the integral and integrating by parts:

$$\int_{-1}^{1} x^m \left[(x^2-1)^n\right]^{(n)} dx = \underbrace{x^m \left[(x^2-1)^n\right]^{(n-1)} \Big|_{-1}^{1}}_{\text{vanishes at } -1 \text{ and } 1.} - m\int_{-1}^{1} x^{m-1} \left[(x^2-1)^n\right]^{(n-1)} dx$$

$$= -m \int_{-1}^{1} x^{m-1}\left[(x^2-1)^n\right]^{(n-1)} dx \quad \Big\} \text{ Integrating by parts again}$$

$$= \underbrace{-m\, x^{m-1}\left[(x^2-1)^n\right]^{(n-2)}\Big|_{-1}^{1}}_{\text{again vanishes}} + m(m-1)\int_{-1}^{1} x^{m-2}\left[(x^2-1)^n\right]^{(n-2)} dx$$

Integrating by parts $(m-2)$ more times and taking into account the fact that the extra-integral term vanishes at -1 and 1, we arrive at:

$$(-1)^m m(m-1)(m-2)\cdot\ldots\cdot 1 \int_{-1}^{1} 1\cdot\left[(x^2-1)^n\right]^{(n-m)} dx \quad \Big\} \begin{array}{l}\text{Integrating this derivative}\\ \text{one more time}\\ (n-m>0)\end{array}$$

$$= (-1)^m m! \left[(x^2-1)^n\right]^{(n-m-1)}\Big|_{-1}^{1} = 0.$$

In the above, we used: $\left[(x^2-1)^n\right]^{(k)} = 0$ for $x=-1$ or $x=1$ for $k = n-1, n-2, \ldots, n-m-1$.

c) $P_m(x) = \sum_{k=0}^{m} a_k x^k$ - polynomial of degree m $(m<n)$.

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \int_{-1}^{1}\sum_{k=0}^{m} a_k x^k P_n(x)\,dx = \sum_{k=0}^{m} a_k \underbrace{\int_{-1}^{1} x^k P_n(x)\,dx}_{=0}$$

$m<n$, so $k<n$ too (from (b)).

$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0.$$

③ (a) $T_n(x) = \cos(n \arccos x)$

$$\int_{-1}^{1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx \qquad m \neq n$$

$$= \int_{-1}^{1} \cos(m \arccos x) \cos(n \arccos x) \frac{dx}{\sqrt{1-x^2}}$$

Introduce new variable : $x = \cos\theta$

$\arccos x = \theta$

$x = 1$, i.e. $\cos\theta = 1 \Rightarrow \theta = 0$ (new integration limits)

$x = 1$, i.e. $\cos\theta = -1 \Rightarrow \theta = \pi$

$$\theta = \arccos x$$

$$d\theta = -\frac{1}{\sqrt{1-x^2}}\, dx .$$

Therefore, the above integral becomes:

$$\int_{\pi}^{0} \cos(m\theta)\cos(n\theta)(-d\theta) = \int_{0}^{\pi} \cos m\theta \cos n\theta \, d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi} \left(\cos(m+n)\theta + \cos(m-n)\theta\right) d\theta$$

$$= \frac{1}{2} \left[\frac{1}{m+n} \sin(m+n)\theta + \frac{1}{m-n} \sin(m-n)\theta \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{1}{m+n} \Big(\underbrace{\sin(m+n)\pi}_{=0} - \underbrace{\sin 0}_{=0} \Big) + \frac{1}{m-n} \Big(\underbrace{\sin(m-n)\pi}_{=0} - \underbrace{\sin 0}_{=0} \Big) \right]$$

$= 0.$

note: $m \neq n$.

The norm:

$$\|T_n\|^2 = \int_{-1}^{1} T_n^2(x)\,\frac{dx}{\sqrt{1-x^2}}$$ Using the same variable substitution as above

$$= \int_0^{\pi} \cos^2(n\theta)\,d\theta$$ $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

$$= \frac{1}{2}\int_0^{\pi} (1+\cos 2n\theta)\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi} d\theta + \int_0^{\pi} \cos 2n\theta\,d\theta\right]$$

$$= \frac{1}{2}\left[\pi + \underbrace{\frac{1}{2n}\sin 2n\theta\Big|_0^{\pi}}_{=0}\right]$$

$$= \frac{\pi}{2}$$

$$\Rightarrow \quad \underline{\|T_n\| = \sqrt{\frac{\pi}{2}}}\,.$$

(b) $T_n(\cos\theta) = \cos(n\arccos(\cos\theta)) = \cos n\theta$

$$= \operatorname{Re} e^{in\theta}$$ $e^{in\theta} = \cos n\theta + i\sin n\theta$ We take the real part.

$$= \operatorname{Re}(e^{i\theta})^n$$

$$= \operatorname{Re}(\cos\theta + i\sin\theta)^n$$ Using the binomial expansion

$$= \operatorname{Re}\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}\, i^k \sin^k\theta$$

$$= \sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(\cos\theta)^{n-2m}(-1)^m \sin^{2m}\theta$$

$$= \sum_{m=0}^{[n/2]}\binom{n}{2m}(\cos\theta)^{n-2m}(-1)^m(\sin^2\theta)^m$$

$$= \sum_{m=0}^{[n/2]}\binom{n}{2m}(-1)^m(\cos\theta)^{n-2m}(1-\cos^2\theta)^m$$

For odd k i^k is <u>imaginary</u>. For even k $i^k = (-1)^{k/2}$, Only even $k=2m$ contribute; from $m=0$ to $m=[\frac{n}{2}]$, where $[\frac{n}{2}]$ integer part of $\frac{n}{2}$.

polynomial in $\cos\theta$.

(4) (a) $v_0 = 1, \quad v_1 = x, \quad v_2 = x^2, \ \ldots$

1) $\|v_0\|^2 = \int_0^\infty 1 \cdot e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -0+1 = 1.$

So, $\|v_0\| = 1$ and we set $\underline{\varphi_0(x)} = v_0 = \underline{1.}$

2) $\tilde{v}_1 = v_1 + C\varphi_0$ should be made orthogonal to φ_1 :

$(\varphi_0, \tilde{v}_1) = 0$ gives $(\varphi_0, v_1) + C\underbrace{(\varphi_0, \varphi_0)}_{=1} = 0$

$\Rightarrow \quad C = -(\varphi_0, v_1)$

$$= -\int_0^\infty 1 \cdot x \cdot e^{-x}\,dx = -\int_0^\infty x e^{-x}\,dx$$

[Using $\int_0^\infty x^n e^{-x}\,dx = n!$]

$$= -1$$

$\Rightarrow \quad \tilde{v}_1 = x - 1$

$$\|\tilde{v}_1\|^2 = \int_0^\infty (x-1)^2 e^{-x}\,dx$$

$$= \int_0^\infty (x^2 - 2x + 1) e^{-x}\,dx$$

$$= 2 - 2 + 1 \quad \left.\right\} \text{Using the integral given above}$$

$$= 1$$

$\Rightarrow \quad \underline{\varphi_1(x)} = \dfrac{\tilde{v}_1}{\|\tilde{v}_1\|} = \underline{x - 1}.$

3) Similarly,

$\tilde{v}_2 = v_2 + C_1\varphi_0 + C_2\varphi_1$

$(\varphi_0, \tilde{v}_2) = 0$ gives $C_1 = -(\varphi_0, v_2)$

$(\varphi_1, \tilde{v}_2) = 0$ gives $C_2 = -(\varphi_1, v_2)$

Hence, $\tilde{v}_2 = v_2 - (\varphi_0, v_2)\varphi_0 - (\varphi_1, v_2)\varphi_1$

$$(\varphi_0, v_2) = \int_0^\infty 1 \cdot x^2 \cdot e^{-x} dx = 2$$

$$(\varphi_1, v_2) = \int_0^\infty (x-1)\, x^2 e^{-x} dx = \int_0^\infty (x^3 - x^2) e^{-x} dx$$

$$= 3! - 2! = 4 .$$

So, we have:

$$\tilde{v}_2 = x^2 - 2 \cdot 1 - 4(x-1)$$

$$= x^2 - 4x + 2$$

Checking the norm:

$$\|\tilde{v}_2\|^2 = \int_0^\infty (x^2 - 4x + 2)^2 e^{-x} dx$$

$$= \int_0^\infty (x^4 + 16x^2 + 4 - 8x^3 + 4x^2 - 16x) e^{-x} dx$$

$$= \int_0^\infty (x^4 - 8x^3 + 20x^2 - 16x + 4) e^{-x} dx$$

$$= 4! - 8 \cdot 3! + 20 \cdot 2! - 16 \cdot 1 + 4$$

$$= 24 - 48 + 40 - 16 + 4$$

$$= 4$$

$$\Rightarrow \|\tilde{v}_2\| = 2 ,$$

$$\underline{\varphi_2(x)} = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \underline{\frac{1}{2}(x^2 - 4x + 2)} .$$

(b) $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$ is a polynomial of degree n. The highest term is obtained when the derivative operates on e^{-x} n times, so

$L_n(x) = (-1)^n x^n + \ldots$ (We will need this later)

$$(x^m, L_n) = \int_0^\infty x^m e^{-x} L_n(x)\, dx$$

$$= \int_0^\infty x^m e^{-x} e^{x} \frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$= \int_0^\infty x^m \frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

Integrating by parts

$$= \underbrace{x^m \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\Big|_0^\infty}_{=0} - m\int_0^\infty x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\, dx$$

$$= \underbrace{-m\, x^{m-1} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\Big|_0^\infty}_{=0} + m(m-1)\int_0^\infty x^{m-2} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\, dx$$

$= \ldots\ldots$ (integrating by parts $m-2$ times)

$$= (-1)^m\, m(m-1)\ldots 2\cdot 1 \int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})\, dx$$

Note: $n-m>0$

$$= (-1)^m\, m!\, \frac{d^{n-m-1}}{dx^{n-m-1}}(x^n e^{-x})\Big|_0^\infty = \underline{\underline{0}}.$$

$$\|L_n^2\|^2 = \int_0^\infty e^{-x} L_n^2(x)\, dx = \int_0^\infty e^{-x}\big((-1)^n x^n + \underbrace{\ldots}_{\text{terms with smaller powers of } x}\big) L_n(x)\, dx$$

$$= (-1)^n \int_0^\infty x^n e^{-x} L_n(x)\, dx$$

Terms with lower powers of x vanish, as shown above

$$= (-1)^n \int_0^\infty x^n \frac{d^n}{dx^n}(x^n e^{-x})\, dx.$$

Integrating by parts n times, we obtain:

$$(-1)^n (-1)^n n! \int_0^\infty x^n e^{-x}\, dx$$

$$= n!\, n! = (n!)^2 .$$

Hence $\|L_n\| = n!$

This proves that <u>normalised</u> polynomials orthogonal with the weight function e^{-x} are

$$\varphi_n(x) = (-1)^n \frac{L_n(x)}{n!} .$$

(The $(-1)^n$ factor is included here to make the coefficient at the x^n term <u>positive</u>.)