AMA3006 Problem sheet 8

Examples

1 1] $(1-x^2)u'' - 2xu' + \lambda u = 0 \qquad (1)$

To be self-adjoint, the equation should have the form

$$(pu')' - qu + \lambda \rho u = 0,$$

or $\quad pu'' + p'u' - qu + \lambda \rho u = 0.$

Note: $\quad (1-x^2)' = -2x$, so Eq. (1) is self-adjoint. It can immediately be written as

$$(\underbrace{(1-x^2)}_{p(x)} u')' + \lambda u = 0$$

$q = 0$, $\underline{\rho = 1}$ is the weight function

2] $$x u'' + \underbrace{(1-x)}_{\text{this is not equal to } x'} u' + \lambda u = 0 \qquad (2)$$

To transform into the self-adjoint form, multiply (2) by $R(x)$:

$$R(x)\, x\, u'' + R(x)(1-x)u' + \lambda R(x) u = 0.$$

We require that

$$(R(x)\,x)' = R(x)(1-x)$$

$$R'x + \cancel{R} = \cancel{R} - Rx$$

$$R'x = -Rx$$

$$\frac{dR}{dx} = -R$$

$$\int \frac{dR}{R} = -\int dx$$

$$\ln R = -x \Rightarrow \underline{R = e^{-x}}.$$

[Note: when finding R, we can choose the arbitrary constant in the indefinite integrals arbitrarily, e.g. equal to zero.]

Multiplying (2) by $R(x) = e^{-x}$, we have:

$$e^{-x} x u'' + e^{-x}(1-x)u' + \lambda e^{-x} u = 0$$

$$(\underbrace{e^{-x} x}_{p(x)} u')' + \lambda e^{-x} u = 0 .$$

$q(x) = 0$ $\quad$ $\underline{\rho(x) = e^{-x}}$ is the weight function

(2) (a) $\quad u = \sum_{k=0}^{\infty} a_k x^k$

$$u' = \sum_{k=0}^{\infty} a_k k x^{k-1}$$

$$u'' = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}$$

Note: $k=0$ and $k=1$ terms are zero in this sum.

Substituting this into

$$(1-x^2)u'' - 2xu' + \lambda u = 0,$$

we obtain:

$$\sum_{k=0}^{\infty} (1-x^2) a_k k(k-1) x^{k-2} - \sum_{k=0}^{\infty} 2x a_k k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\underbrace{\sum_{k=}^{\infty} a_k k(k-1) x^{k-2}}_{} - \sum_{k=0}^{\infty} a_k k(k-1) x^k - \sum_{k=0}^{\infty} 2a_k k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

In this sum, replace:

$k-2 \to k$

$k-1 \to k+1$

$k \to k+2$

After this, all sums contain x^k and we can write the above expression as a single sum, and factorise out x^k:

$$\sum_{k=0}^{\infty} \left[a_{k+2}(k+2)(k+1) - a_k k(k-1) - 2a_k k + \lambda a_k \right] x^k = 0 .$$

Hence, the expression in square brackets must be zero:

$$a_{k+2}(k+1)(k+2) - a_k k(k+1) + \lambda a_k = 0$$

$$\Rightarrow \quad a_{k+2} = -\frac{\lambda - k(k+1)}{(k+1)(k+2)} a_k \, , \qquad (1)$$

the reccurrence relation for the coefficients.

If we start from a_0, it gives $a_2, a_4, \ldots$

If we start from $a_1 \neq 0$, it gives $a_3, a_5, \ldots$

(b) For the sum $\sum_{k=0}^{\infty} a_k x^k$ to be a polynomial, all coefficients starting from a certain highes nonzero a_n should vanish. This can be achieved if $\lambda = n(n+1)$, where n is a non-negative integer, $n = 0, 1, 2, \ldots$

In this case $a_{n+2} = 0$, and so are all higher a_k. The choice of the smallest nonzero coefficient (a_0 for even n and a_1 for odd n) is a matter of convention.

Note that if $\lambda \neq n(n+1)$ then the series is infinite. For large k, $k(k+1) \gg \lambda$, and (1) becomes:

$$a_{k+2} \approx \frac{k\cancel{(k+1)}}{\cancel{(k+1)}(k+2)} a_k$$

$$\approx a_k \qquad \left(k \gg 1, \text{ so that } \frac{k}{k+2} \approx 1 \right).$$

Hence, the higher k part of the series behaves as:

$$u = \text{const}\left(\dots + x^{k} + x^{k+2} + x^{k+4} + \dots \right),$$

i.e. behaves as a geometric series.

Its sum is proportional to $\dfrac{1}{1-x^2}$,

so that $\quad u \simeq \dfrac{1}{1-x^2}$.

Therefore, such solutions diverge at the ends of the interval $[-1, 1]$.

(Since $p = 1-x^2$ vanishes at $x = \pm 1$, we are dealing with a singular Sturm–Liouville problem.) Hence, by requiring that the solution is <u>finite</u> we select $\lambda = n(n+1)$.

The polynomials obtained in this way are orthogonal to each other (with the weight $\rho = 1$) on the interval $[-1, 1]$ *), and are in fact identical to the <u>Legendre polynomials</u> (if we choose a_0 or a_1 in a suitable way), for which

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad (m \neq n).$$

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*) This is a property of the eigenfunctions of the Sturm–Liouville problem for the equation

$$((1-x^2)u')' + \lambda u = 0.$$

(c) According to Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}.$$

Let us verify that this polynomial satisfies

$$(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0 \quad . (*)$$

Consider $\quad v = (x^2-1)^n, \qquad P_n(x) = \frac{1}{2^n n!} v^{(n)}.$

$$v' = n(x^2-1)^{n-1} 2x$$

$$(x^2-1) v' = n(x^2-1)^n 2x$$

$$(x^2-1) v' = 2xnv$$

Note: $P_n(x)$ and $v^{(n)}$ satisfies the same equation

$v^{(n)}$ - nth derivative of v.

Differentiating this equation:

$$2x v' + (x^2-1) v'' = 2nv + 2xnv'$$

$$(x^2-1) v'' = 2nv + 2x(n-1)v'$$

Differentiating again:

$$2x v'' + (x^2-1) v''' = 2nv' + 2(n-1)v' + 2x(n-1)v''$$

$(k=1)$ $\qquad (x^2-1) v''' = (4n-2)v' + 2x(n-2)v''$

And again:

$$2x v''' + (x^2-1) v^{IV} = (4n-2)v'' + 2(n-2)v'' + 2x(n-2)v'''$$

$(k=2)$ $\qquad (x^2-1) v^{IV} = (6n-6)v'' + 2x(n-3)v'''$

Noticing the pattern, we see that after $k-2$ more differentiations, we will obtain:

$$(x^2-1) v^{(k+2)} = [2(k+1)n - k(k+1)] v^{(k)} + 2x(n-k-1) v^{(k+1)}$$

For $k=n$ this gives:

$$(x^2-1) v^{(n+2)} = [2(n+1)n - n(n+1)] v^{(n)} + 2x(n-n-1) v^{(n+1)}$$

or: $\qquad (x^2-1)(v^{(n)})'' + 2x(v^{(n)})' - n(n+1) v^{(n)} = 0,$

and multiplying by -1, we obtain Eq. (*), as required

3 (a) $$xu'' + (1-x)u' + \lambda u = 0$$

$$u = \sum_{k=0}^{\infty} a_k x^k, \quad u' = \sum_{k=0}^{\infty} a_k k x^{k-1}, \quad u'' = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}$$

Substituting into the equation:

$$\sum_{k=0}^{\infty} a_k k(k-1) x^{k-1} + \sum_{k=0}^{\infty} (1-x) a_k k x^{k-1} + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$$\underbrace{\sum_{k=0}^{\infty} a_k k(k-1) x^{k-1} + \sum_{k=0}^{\infty} a_k k x^{k-1}}_{\text{joining these}} - \sum_{k=0}^{\infty} a_k k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$$\underbrace{\sum_{k=0}^{\infty} a_k k^2 x^{k-1}} - \sum_{k=0}^{\infty} a_k k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

Replace: $k-1 \to k$
$k \to k+1$

$$\sum_{k=0}^{\infty} \big(\underbrace{a_{k+1}(k+1)^2 - a_k k + \lambda a_k}_{\text{must be equal to zero}} \big) x^k = 0 .$$

$$\Rightarrow \quad a_{k+1} = -\frac{\lambda - k}{(k+1)^2} a_k .$$

Starting from $k = 0$, $a_0 \neq 0$, we can use the recurrence relation to generate $a_1, a_2, \ldots$

(b) The solution will be a polynomial if the coefficients vanish, starting from a certain point. This will happen if $\lambda = n$, where n is a non-negative integer, $n = 0, 1, 2, \ldots$ Then a_n will be the highest nonzero coefficients, and $a_{n+1} = a_{n+2} = \ldots = 0$.

What if $\lambda \neq n$? The set of coefficients will be infinite. For $k \gg \lambda$, the recurrence relation takes the form:

$$a_{k+1} \approx \frac{k}{(k+1)^2} a_k$$

$$\approx \frac{a_k}{k+1} \quad \Bigg\} \quad \frac{k}{k+1} \approx 1 \text{ for } k \gg 1.$$

Hence, for high k the series has the form:

$$u \approx \text{const} \left[\dots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} + \frac{x^{k+2}}{(k+2)!} + \dots \right]$$

(where we wrote $a_k = \frac{\text{const}}{k!}$, for convenience).

In this series we recognise the same pattern as in $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Hence, the solution for $\lambda \neq n$ behaves as $u \simeq e^x$, i.e. diverges strongly at $x \to \infty$.

From the self-adjoint form of the equation (see ①),

$$(x e^{-x} u')' + \lambda e^{-x} u = 0,$$

we notice that the solutions (eigenfunctions) must be orthogonal with the weight function e^{-x} on $[0, +\infty)$, $\int_0^\infty u_m(x) u_n(x) e^{-x} dx = 0 \quad (m \neq n)$

Clearly, this integral would <u>diverge</u> if the functions behaved as $u \sim e^x$. Hence, integer values only, $\lambda = n$, $n = 0, 1, 2, \dots$ must be used.