Chapter 7

Continuous Time Processes

7.1 Introduction

In a continuous time stochastic process (with discrete state space), a change of state can occur at any time instant. The associated *point process* consists of those instants

$$(t_0 = 0), t_1, t_2, ...$$

at which a change of state (or transition) occurs. The intervals

$$g_n = t_n - t_{n-1}, \qquad n \geq 1$$

are called *gaps* or *sojourn times*. A point process can be specified by giving a (probabilistic) mechanism for determining *either* the times $t_1, t_2, ...$ *or* the gaps $g_1, g_2, ...$ To specify the continuous time process we must also give a mechanism for determining the transitions at $t_1, t_2, ...$

7.2 Counting Processes

Such processes are particularly simple: the state variable is just a *count* which increases by 1 at each transition point t_i. A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of transitions or events that have occurred in $(0, t]$. $N(t)$ must satisfy the following conditions:

(i) $N(t)$ is a count r.v., with $N(0) = 0$.

(ii) If $s < t$, then $N(s) \leq N(t)$.

(iii) For $s < t$, $N(t) - N(s)$ is equal to the number of events that have occurred in $(s, t]$.

A counting process is said to possess

(i) *independent increments* if the number of events which occur in disjoint (i.e. non-overlapping) time intervals are independent;

(ii) *stationary* or *time-homogeneous increments* if the probability distribution of the number of events in the interval $(u, u + t]$ $(u \geq 0)$ (i.e. $N(u + t) - N(t)$) depends only on t, i.e. on the *length* of the time interval and not on its *position* on the time axis.

Let the r.v. T_n denote the time at which the n^{th} event after $t = 0$ occurs, i.e.

$$T_n = \inf\{t : N(t) = n\}, \qquad n \geq 1,$$

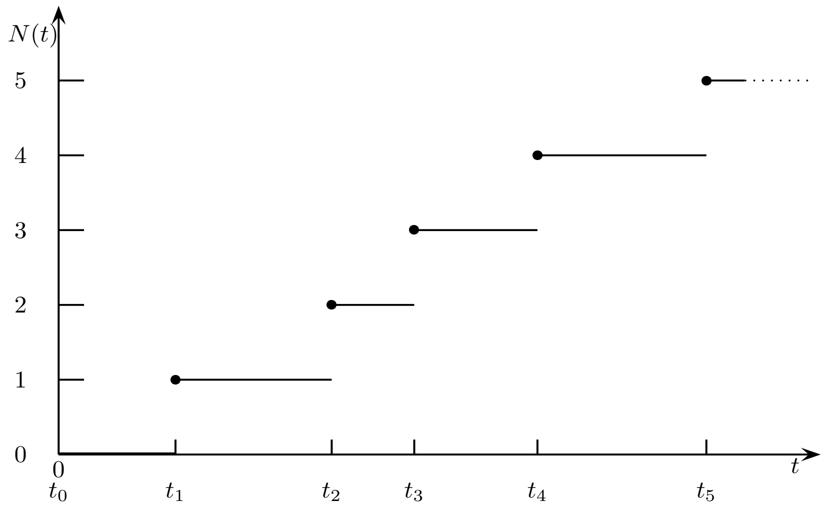
and let $T_0 = 0$. Note that

$$T_0 \leq T_1 \leq T_2 \leq \cdots$$

and

$$N(t) = \max\{n : T_n \leq t\}$$

A typical realization of a counting process is shown.



7.3 Poisson Process

7.3.1 Definition and distribution of $N(t)$

The counting process $\{N(t), t \geq 0\}$ is said to be a *Poisson process* having rate λ if

(i) the process has independent increments;

(ii) for any short time interval $(t, t + \delta t]$,

$$\text{P}(1 \text{ event occurs in } (t, t + \delta t]) = \lambda\delta t + o(\delta t) \quad (\text{as } \delta t \to 0),$$

i.e.

$$\text{P}(N(t + \delta t) - N(t) = 1) = \lambda\delta t + o(\delta t)$$

or, to put it another way,

$$\text{P}(N(t + \delta t) = n + 1 | N(t) = n) = \lambda\delta t + o(\delta t), \quad n \geq 0;$$

(iii) $$\text{P}(k \text{ events occur in } (t, t + \delta t]) = o(\delta t), \quad k \geq 2,$$

or, alternatively,

$$\text{P}(2 \text{ or more events occur in } (t + \delta t]) = o(\delta t),$$

each of which properties can be similarly expressed in terms of $N(t)$.

Theorem

If $\{N(t), t > 0\}$ is a Poisson process, then, for each $t > 0$, the r.v. $N(t)$ has the Poisson distribution with parameter λt, i.e.

$$\boxed{\mathrm{P}(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n = 0, 1, 2, ... \qquad (7.1)}$$

Proof: For convenience write $p_n(t)$ for $\mathrm{P}(N(t) = n)$. Since $N(0) = 0$, we have that

$$p_0(0) = 1, \qquad p_n(0) = 0 \text{ for } n > 0. \tag{7.2}$$

Now, invoking independence,

$$p_n(t + \delta t) = \mathrm{P}(n \text{ events in } (0, t + \delta t]) = \sum_{k=0}^{n} \mathrm{P}(n - k \text{ events in } (t, t + \delta t]).\mathrm{P}(k \text{ events in } (0, t]).$$

But since

$$\begin{aligned} & \mathrm{P}(0 \text{ events occur in } (t, t + \delta t]) \\ + \; & \mathrm{P}(1 \text{ event occurs in } (t, t + \delta t]) \\ + \; & \mathrm{P}(2 \text{ or more events occur in } (t, t + \delta t]) = 1, \end{aligned}$$

we have that

$$\mathrm{P}(0 \text{ events occur in } (t, t + \delta t]) = 1 - \lambda \delta t + o(\delta t).$$

Then

$$\begin{aligned} p_n(t + \delta t) &= (1 - \lambda \delta t + o(\delta t)).p_n(t) \\ &\quad +(\lambda \delta t + o(\delta t)).p_{n-1}(t) + o(\delta t), \quad n \geq 1 \\ \text{i.e.} \quad p_n(t + \delta t) &= (1 - \lambda \delta t) p_n(t) + \lambda \delta t p_{n-1}(t) + o(\delta t), \quad n \geq 1. \end{aligned}$$

Also

$$\begin{aligned} p_0(t + \delta t) &= p_0(t).(1 - \lambda \delta t + o(\delta t)) \\ &= (1 - \lambda \delta t) p_0(t) + o(\delta t). \end{aligned}$$

So

$$\begin{aligned} \frac{p_n(t + \delta t) - p_n(t)}{\delta t} &= \lambda p_{n-1}(t) - \lambda p_n(t) + \frac{o(\delta t)}{\delta t}, \quad n \geq 1 \\ \text{and} \quad \frac{p_0(t + \delta t) - p_0(t)}{\delta t} &= -\lambda p_0(t) + \frac{o(\delta t)}{\delta t}. \end{aligned}$$

Now let $\delta t \to 0$: we obtain

$$\boxed{\begin{aligned} \frac{dp_n(t)}{dt} &= \lambda p_{n-1}(t) - \lambda p_n(t), \qquad n \geq 1 \\ \frac{dp_0(t)}{dt} &= -\lambda p_0(t). \end{aligned} \qquad (7.3)}$$

Using the initial conditions (7.2), this set of difference-differential equations can be solved successively for $n = 0, 1, 2, ...$ to give

$$p_0(t) = e^{-\lambda t}, \quad p_1(t) = \lambda t e^{-\lambda t}, \quad ....$$

and hence by induction to give the general result (7.1) for $p_n(t)$.

Alternatively, we can use PGFs to solve (7.3). Let $G(s, t)$ be the PGF of $\{p_n(t), n \geq 0\}$, i.e.

$$\boxed{G(s, t) = \sum_{n=0}^{\infty} p_n(t) s^n \qquad (7.4)}$$

(for fixed $t \geq 0$, $G(s, t)$ has the usual PGF properties).

Multiply (7.1) by s^n and sum over $n = 0, 1, 2, ...$ to get

$$\frac{dp_0(t)}{dt} s^0 + \sum_{n=1}^{\infty} \frac{dp_n(t)}{dt} s^n = -\lambda p_0(t) s^0 + \sum_{n=1}^{\infty} \lambda \{p_{n-1}(t) - p_n(t)\} s^n$$

i.e.

$$\begin{aligned}\frac{\partial G(s,t)}{\partial t} &= \sum_{n=0}^{\infty} \frac{dp_n(t)}{dt} s^n \\ &= -\lambda \sum_{n=0}^{\infty} p_n(t)s^n + \lambda \sum_{n=1}^{\infty} p_{n-1}(t)s^n = -\lambda G(s,t) + \lambda s G(s,t).\end{aligned}$$

Thus

$$\frac{1}{G}.\frac{\partial G}{\partial t} = \lambda(s-1)$$

$$\text{i.e. } \frac{\partial}{\partial t}\{\log_e G(s,t)\} = \lambda(s-1)$$

$$\text{i.e. } \log G(s,t) = \lambda(s-1)t + C(s)$$

where $C(s)$ is constant with respect to t. The initial conditions (7.2) give $G(s,0) = 1$, so $0 = 0 + C(s)$, i.e. $C(s) = 0$. It follows that

$$\boxed{G(s,t) = \exp\{\lambda t(s-1)\},} \qquad (7.5)$$

which is the PGF of the Poisson distribution with parameter λt – hence the result. □

Corollary

$$\boxed{\mathrm{P}[N(u+t) - N(u) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n \geq 0, \text{ for } u \geq 0, t > 0} \qquad (7.6)$$

i.e., the Poisson process has stationary increments (as we would expect from conditions (ii) and (iii)).

The **proof** proceeds along the same lines (let $p_n(u,t)$ be the above probability).

7.3.2 Other associated distributions

The *interarrival* (or *inter-event*) *times* in a Poisson process are the r.v.s $X_1, X_2, ...$ given by

$$\boxed{X_n = T_n - T_{n-1}, \qquad n \geq 1.} \qquad (7.7)$$

Then

$$\mathrm{P}(X_1 > x) = \mathrm{P}(N(x) = 0) = e^{-\lambda x}, \qquad x \geq 0$$

so the c.d.f. of X_1 is $1 - e^{-\lambda x}$, i.e.

$$X_1 \sim \exp(\lambda).$$

Also

$$\mathrm{P}(X_2 > x | X_1 = x_1) = \mathrm{P}(\text{no arrival in } (x_1, x_1 + x] | X_1 = x_1).$$

In this conditional probability, the first event concerns arrivals in $(x_1, x_1 + x]$ and the second (conditioning) event concerns arrivals in $[0, x_1]$, so the events are independent. Hence

$$\mathrm{P}(X_2 > x | X_1 = x_1) = \mathrm{P}(\text{no arrival in } (x_1, x_1 + x]) = e^{-\lambda x}$$

(using the Corollary); i.e., X_2 is independent of X_1 and has the same distribution. Similarly

$$\mathrm{P}(X_{n+1} > x | X_1 = x_1, ..., X_N = x_n) = \mathrm{P}(\text{no arrival in } (t, t+x])$$

where $t = x_1 + x_2 + \cdots + x_n$. It follows by induction on n that $X_1, X_2, X_3, ...$ are independent exponential r.v.s, each with parameter λ.

The *waiting time* for the r^{th} event is the r.v.

$$W_r = X_1 + X_2 + \cdots + X_r, \tag{7.8}$$

and so

$$W_r \sim \text{Gamma}(r, \lambda). \tag{7.8}$$

An alternative derivation of this result is as follows.

$$W_r \leq w \Longleftrightarrow N(w) \geq r.$$

So

$$\begin{aligned} F_{W_r}(w) &= \text{P}(W_r \leq w) \\ &= \text{P}(N(w) \geq r) = \sum_{j=r}^{\infty} \frac{(\lambda w)^j e^{-\lambda w}}{j!}. \end{aligned}$$

Differentiating to get the p.d.f., we again find that $W_r \sim \text{Gamma}(r, \lambda)$.

More generally, this is also the distribution of time between the m^{th} and $(m+r)^{\text{th}}$ events.

7.4 Markov Processes

A discrete-state continuous-time stochastic process $\{X(t), t \geq 0\}$ is said to be a *Markov process* if

$$\text{P}(X(t_n) = j | X(t_0) = i_0, X(t_1) = i_1, ..., X(t_{n-1}) = i) = \text{P}(X(t_n) = j | X(t_{n-1}) = i) \tag{7.9}$$

for all $t_0 < t_1 < \cdots < t_n$ and all relevant values of $i, j, i_0, i_1, ...$

A Markov process is said to be time-homogeneous or *stationary* if

$$\text{P}(X(u+t) = j | X(u) = i) = \text{P}(X(t) = j | X(0) = i) = p_{ij}(t) \tag{7.10}$$

for all i, j and $u > 0, t > 0$.

A stationary Markov process is completely described by its *transition probability functions* $\{p_{ij}(t)\}$ and its initial probability distribution.

In modelling Markov processes it is generally assumed that

$$p_{ij}(\delta t) = \begin{cases} \lambda_{ij}\delta t + o(\delta t) & i \neq j \\ 1 + \lambda_{ii}\delta t + o(\delta t), & i = j, \end{cases} \tag{7.11}$$

where the constants λ_{ij} are called *transition rates* and satisfy

$$\begin{aligned} \lambda_{ij} &\geq 0, \quad i \neq j, \\ \lambda_{ii} &< 0, \quad \text{all } i, \\ \text{and} \quad \sum\nolimits_j \lambda_{ij} &= 0 \quad (\text{since } \sum\nolimits_j p_{ij}(\delta t) = 1). \end{aligned} \tag{7.12}$$

<u>Example</u> The Poisson process with rate λ is a simple example of a Markov process, with transition rates

$$\lambda_{n,n+1} = \lambda, \quad \lambda_{n,n} = -\lambda, \quad \lambda_{n,m} = 0, \ m \neq n, n+1.$$

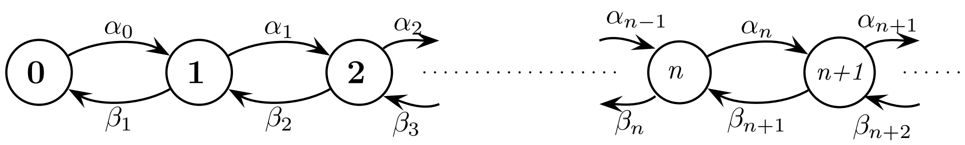
We do not pursue the theory of general Markov processes in this module: instead we study a particular Markov process from first principles.

7.5 Birth-and-Death Process

Here $X(t)$ may be thought of as the size of a 'population' at time t (with possible values $0, 1, 2, ...$), with 'births' and 'deaths' occurring from time to time in such a way that

$$p_{ij}(\delta t) = \begin{cases} \alpha_i \delta t + o(\delta t) & i \geq 0, j = i+1 \\ \beta_i \delta t + o(\delta t) & i \geq 1, j = i-1 \\ 1 - (\alpha_i + \beta_i)\delta t + o(\delta t) & i \geq 0, j = i \\ o(\delta t) & \text{otherwise.} \end{cases} \tag{7.13}$$

The process is conveniently represented on a *transition rate diagram*, as shown.



(The nodes represent possible states $X(t)$ and the arrows possible transitions to other states. Note that the transition rates are in general state-dependent).

For convenience, we again write $p_n(t) = \mathrm{P}(X(t) = n), \quad n = 0, 1, 2, ...$

Then

$$\mathrm{P}(X(t+\delta t) = n) = \sum_{i=0}^{\infty} \mathrm{P}(X(t+\delta t) = n | X(t) = i).\mathrm{P}(X(t) = i)$$

i.e.

$$p_n(t+\delta t) = \sum_{i=0}^{\infty} p_{in}(\delta t) p_i(t).$$

Inserting (7.13)) gives

$$\begin{aligned} p_n(t+\delta t) &= (1 - \alpha_n \delta t - \beta_n \delta t) p_n(t) \\ &\quad + (\alpha_{n-1}\delta t) p_{n-1}(t) \\ &\quad + (\beta_{n+1}\delta t) p_{n+1}(t) + o(\delta t), \quad n \geq 1 \\ \text{and} \quad p_0(t+\delta t) &= (1 - \alpha_0 \delta t) p_0(t) + (\beta_1 \delta t) p_1(t) + o(\delta t). \end{aligned}$$

Then

$$\begin{aligned} \frac{dp_n(t)}{dt} &= \lim_{\delta t \to 0} \frac{p_n(t+\delta t) - p_n(t)}{\delta t} \\ &= -(\alpha_n + \beta_n) p_n(t) + \alpha_{n-1} p_{n-1}(t) + \beta_{n+1} p_{n+1}(t), n \geq 1 \\ \text{and} \quad \frac{dp_0(t)}{dt} &= -\alpha_0 p_0(t) + \beta_1 p_1(t). \end{aligned} \tag{7.14}$$

We shall solve this set of difference-differential equations in a special case, again using the PGF.

Example **Linear birth-and-death process**

Suppose that

$$\boxed{\begin{array}{rcll} \alpha_n & = & n\alpha, & n \geq 0 \\ \beta_n & = & n\beta, & n \geq 1. \end{array}} \tag{7.15}$$

(*Note*: since $\alpha_0 = 0$, $X(t) = 0$ is absorbing.)

Suppose that initially

$$X(0) = a \quad (> 0)$$

i.e.

$$\begin{array}{rcll} p_a(0) & = & 1 & \\ p_m(0) & = & 0, & m \neq a. \end{array}$$

The PGF of $X(t)$ is

$$\boxed{G(s,t) = \sum_{n=0}^{\infty} p_n(t) s^n} \tag{7.16}$$

where $G(s,0) = s^a$.

Multiply (7.14) by s^n and sum over n to get

$$\begin{aligned} \frac{\partial G(s,t)}{\partial t} &= [-(\alpha+\beta)s + \alpha s^2 + \beta] \sum_{n=1}^{\infty} n p_n(t) s^{n-1} \\ &= (\alpha s - \beta)(s-1) \frac{\partial G(s,t)}{\partial s} \end{aligned}$$

It can be verified that the solution satisfying the initial condition $G(s,0) = s^a$ is

$$G(s,t) = \begin{cases} \left[\dfrac{(\alpha s - \beta) e^{(\beta-\alpha)t} - \beta(s-1)}{(\alpha s - \beta) e^{(\beta-\alpha)t} - \alpha(s-1)} \right]^a, & \text{if } \alpha \neq \beta \\ \left[\dfrac{s - \alpha t(s-1)}{1 - \alpha t(s-1)} \right]^a, & \text{if } \alpha = \beta. \end{cases} \tag{7.17}$$

Then $p_n(t)$ is the coefficient of s^n when $G(s,t)$ is expanded as a power series in s.

The *probability of extinction* by time t is

$$p_0(t) = G(0,t) = \begin{cases} \left[\dfrac{\beta - \beta e^{(\beta-\alpha)t}}{\alpha - \beta e^{(\beta-\alpha)t}} \right]^a, & \text{if } \alpha \neq \beta \\ \left[\dfrac{\alpha t}{\alpha t + 1} \right]^a, & \text{if } \alpha = \beta \end{cases}. \tag{7.18}$$

Thus

$$p_0(t) \rightarrow \begin{cases} 1, & \text{if } \alpha \leq \beta \\ (\beta/\alpha)^a, & \text{if } \alpha > \beta \end{cases} \quad \text{as } t \rightarrow \infty; \tag{7.19}$$

i.e., extinction is certain if and only if $\alpha \leq \beta$.

The *mean population size* is

$$\mathrm{E}\{X(t)\} = \left[\frac{\partial G(s,t)}{\partial s} \right]_{s=1} = \begin{cases} a e^{(\alpha-\beta)t} & \text{if } \alpha \neq \beta \\ a & \text{if } \alpha = \beta. \end{cases} \tag{7.20}$$

So, as $t \rightarrow \infty$,

$$\mathrm{E}\{X(t)\} \rightarrow \begin{cases} 0 & \text{if } \alpha < \beta \\ a & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta. \end{cases} \tag{7.21}$$

7.6 Steady-State Distribution

Often it is sufficient to study a Markov process in the limit $t \to \infty$. Under certain conditions

$$p_n(t) \to \pi_n \qquad \text{as } t \to \infty, \tag{7.22}$$

where $\{\pi_n\}$ is *the equilibrium* or *steady-state distribution*, obtained by setting $\dfrac{dp_n}{dt} = 0$, $p_n(t) = \pi_n$ for all n in the appropriate differential equations. So for the birth-and-death process we have

$$\begin{aligned} 0 &= -(\alpha_n + \beta_n)\pi_n + \alpha_{n-1}\pi_{n-1} + \beta_{n+1}\pi_{n+1}, \quad n \geq 1 \\ 0 &= -\alpha_0\pi_0 + \beta_1\pi_1, \end{aligned}$$

or

$$(\alpha_n + \beta_n)\pi_n = \alpha_{n-1}\pi_{n-1} + \beta_{n+1}\pi_{n+1}, \quad n \geq 0, \tag{7.23}$$

where we have defined $\alpha_{-1} = \beta_0 = 0$ for mathematical convenience. Now sum these equations from $n = 0$ to $n = m$ where $m \geq 0$: we obtain

$$\begin{aligned} \sum_{n=0}^{m} (\alpha_n + \beta_n)\pi_n &= \sum_{n=0}^{m} \alpha_{n-1}\pi_{n-1} + \sum_{n=0}^{m} \beta_{n+1}\pi_{n+1} \\ &= \sum_{n=0}^{m-1} \alpha_n\pi_n + \sum_{n=0}^{m+1} \beta_n\pi_n \\ \text{i.e. } \alpha_m\pi_m &= \beta_{m+1}\pi_{m+1}, \qquad m \geq 0, \end{aligned} \tag{7.24}$$

from which we deduce that

$$\boxed{\pi_m = \frac{\alpha_0\alpha_1.....\alpha_{m-1}}{\beta_1\beta_2.....\beta_m}\pi_0, \quad m \geq 1.}$$

The 'normalisation' requirement

$$\sum_{m=0}^{\infty} \pi_m = 1 \tag{7.25}$$

yields

$$\boxed{\pi_0 = S^{-1}, \qquad (7.26)}$$

where

$$\boxed{S = 1 + \frac{\alpha_0}{\beta_1} + \frac{\alpha_0\alpha_1}{\beta_1\beta_2} + ... \qquad (7.27)}$$

The steady-state distribution (if it exists) can be derived directly without appealing to the differential equations. It can be shown that when a general Markov process has a steady-state distribution $(\pi_0, \pi_1, ...) = \boldsymbol{\pi}$, it satisfies the equations

$$\boldsymbol{\pi Q} = \mathbf{0}, \tag{7.28}$$

where

$$\boldsymbol{Q} = (\lambda_{ij}), \quad \mathbf{0} = (0, 0, ...),$$

together with the normalisation condition $\sum_i \pi_i = 1$.

For a birth-and death process

$$\boldsymbol{Q} = \begin{pmatrix} -\alpha_0 & \alpha_0 & 0 & 0 & .. & .. & .. \\ \beta_1 & -(\alpha_1 + \beta_1) & \alpha_1 & 0 & .. & .. & .. \\ 0 & \beta_2 & -(\alpha_2 + \beta_2) & \alpha_2 & .. & .. & .. \\ 0 & 0 & \beta_3 & -(\alpha_3 + \beta_3) & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \end{pmatrix}$$

and substitution in (7.28) yields the equations (7.23) derived above.

7.7 Simple Queueing Systems

7.7.1 Introduction

Queueing is a common human activity and hence it is convenient to use human terminology in discussing queueing systems (although such a system might involve machines waiting for service, messages arriving at a switchboard, etc.). *Customers* arriving at a service point require service from *servers* (channels, counters), but are not always able to get it immediately.

A queueing system is specified by

(i) **The arrival pattern** This is generally described by specifying the probability distribution of the inter-arrival times; these are usually considered to be independent, but may depend e.g. on the queue size.

(ii) **The queue discipline** This determines the order in which customers are served : many everyday queues are 'first-in, first-out'(FIFO). Some queues operate on a priority basis (e.g. in an accident and emergency department of a hospital), and sometimes the system has limited capacity.

(iii) **The service mechanism** The service time (nearly always considered to be independent of the arrival pattern) is usually modelled by some continuous distribution. There may be several servers, or waiting customers may be served in batches (e.g. a queue for a lift or bus).

For relatively simple systems a queue is described by the notation **A / B / s** (due to Kendall), where

A : inter-arrival time distribution,

B : service time distribution,

s : number of servers.

Examples:

(i) The D / M / 1 queue has a constant inter-arrival time (D for deterministic) and an exponential service time distribution (M for Markov), with 1 server.

(ii) The M / M / 2 queue has an exponential inter-arrival time distribution (or equivalently the arrival pattern is a Poisson process) and a (different) exponential service time distribution, with 2 servers.

Amongst the aspects of interest are

- the probability distribution of the number of customers in the queue;
- the distribution of a customer's waiting time (queueing time + service time);
- the busy (or idle) time of the servers.

Much of the simple theory has been developed under the assumption that the queueing system is in the equilibrium or steady state, and we shall confine our discussion to this case.

7.7.2 Queues as birth-and-death processes

Consider a queueing system in which, if the current queue size (that is, number of customers waiting or being served) is n, then

(i) the time to the next *arrival* is exponentially distributed with parameter λ_n;

(ii) the time to the next *departure* (upon completion of service) is exponentially distributed with parameter μ_n.

Then the system can be modelled as a birth-and-death process with

$$\begin{aligned} &X(t) = \text{number of customers in the system at time } t; \\ &\alpha_n = \lambda_n, \qquad \beta_n = \mu_n. \end{aligned}$$

If a steady-state distribution $\{\pi_n\}$ exists,

$$\boxed{\pi_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2,,,,\mu_n}\pi_0, \quad n \geq 1,} \tag{7.24}$$

where

$$\boxed{\pi_0 = S^{-1}, \qquad S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \cdots} \tag{7.25}$$

7.7.3 Specific Queues

(i) **M/M/1 queue ('simple' queue)**

In this case the arrival and service rates are constant:

$$\lambda_n = \lambda, \quad n \geq 0; \qquad \mu_n = \mu, \quad n \geq 1. \tag{7.26}$$

All inter-arrival times have the same negative exponential distribution, with parameter λ, i.e. arrivals follow a Poisson process with parameter λ. There is one server, whose service time is negative exponential with mean μ^{-1}, i.e. service completions follow a Poisson process with parameter μ. Let

$$\rho = \frac{\lambda}{\mu} = \frac{\text{Mean service time}}{\text{Mean inter-arrival time}}$$

(called the *traffic intensity*). Then

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0 = \rho^n \pi_0,$$

where

$$\pi_0 = S^{-1}, \quad S = 1 + \rho + \rho^2 + \cdots = \frac{1}{1-\rho} \text{ provided } 0 \leq \rho < 1.$$

Thus $0 \leq \rho < 1$ is the condition for the existence of a steady-state distribution, and then

$$\boxed{\pi_n = (1-\rho)\rho^n, \quad n \geq 0} \tag{7.27}$$

– a modified geometric distribution with mean $\dfrac{\rho}{1-\rho}$ (being the average queue size).

(ii) Queue with discouragement

This is the situation when the arrival rate decreases as n, the number of customers in the system, increases. Suppose for example that

$$\begin{aligned} \lambda_n &= \frac{\lambda}{n+1}, \qquad n \geq 0 \\ \mu_m &= \mu, \qquad n \geq 1. \end{aligned} \tag{7.28}$$

Then

$$\boxed{\pi_n = \frac{\rho^n}{n!} e^{-\rho}, \quad n \geq 0, \rho = \lambda/\mu.} \tag{7.29}$$

Note that in this case the steady-state distribution exists for *all* $\rho \geq 0$.

(iii) Queue with limited capacity

This can be viewed as a special case of the previous situation: now λ_n is falls abruptly to zero for some critical value of n and above. For example, suppose that we have a 'simple' queue with the additional restriction that there is room in the system for at most N customers. Then we set

$$\lambda_n = \begin{cases} \lambda & \text{for } n < N \\ 0 & \text{for } n \geq N. \end{cases} \tag{7.30}$$

(iv) M/M/s queue

In this case there are s independent servers: suppose that each serves customers at rate μ. Then we set

$$\boxed{\mu_n = \begin{cases} n\mu, & n < s \\ s\mu, & n \geq s. \end{cases}} \tag{7.31}$$

For, so long as all the servers are occupied ($n \geq s$), the service completions comprise s independent Poisson processes and therefore together follow a single Poisson process with rate $s\mu$: otherwise we have n independent Poisson processes, equivalent to a single Poisson process with rate $n\mu$.

(v) M/M/k queue with capacity k

An example of such a system is a telephone exchange with k lines and a 'blocked calls cleared' protocol; i.e. any call which arrives to find all lines being utilised is lost to the system. In this case we take

$$\begin{aligned} \lambda_n &= \begin{cases} \lambda\ , & n < k \\ 0, & n \geq k \end{cases} \\ \mu_n &= \begin{cases} n\mu, & 1 \leq n \leq k \\ 0, & n > k. \end{cases} \end{aligned} \tag{7.32}$$

(vi) Queue with ample independent servers

'Ample' means that a server is *always* available to serve a customer immediately upon arrival, so that there are no *waiting* customers at all! Then, in the simplest case, we have

$$\begin{aligned} \lambda_n &= \lambda, \qquad n \geq 0 \\ \mu_n &= n\mu, \qquad n \geq 1. \end{aligned} \tag{7.33}$$

For further analysis of some of the above cases, see Homework 9.

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