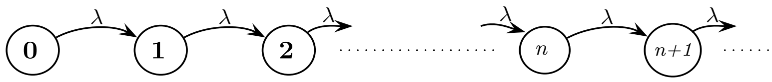
SOR201 Solutions to Examples 9

1. (i) (a)



$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & \ldots\ldots\ldots\ldots & & \\ 0 & -\lambda & \lambda & 0 & \ldots\ldots\ldots & \\ 0 & 0 & -\lambda & \lambda & 0 & \ldots. \\ \ldots\ldots\ldots\ldots & & & & & \end{pmatrix}$$

(b) Since a Poisson process possesses stationary increments,

$$\begin{aligned} \mathrm{P}[N(t_2) - N(t_1) = n] &= \mathrm{P}[N(t_2 - t_1) - N(0) = n] \\ &= \frac{\{\lambda(t_2 - t_1)\}^n \exp\{-\lambda(t_2 - t_1)\}}{n!}, \quad n \geqslant 0 \end{aligned}$$

i.e. the probability depends on the *length* of the time interval, not on its *position* on the time axis.

(c) $F_{W_r}(w) = \mathrm{P}(W_r \leqslant w) = 1 - \displaystyle\sum_{j=0}^{r-1} \frac{(\lambda w)^j e^{-\lambda w}}{j!}$.

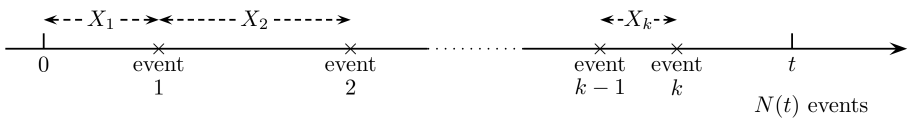
So

$$\begin{aligned} f_{W_r}(w) &= \frac{dF_{W_r}(w)}{dw} \\ &= -\sum_{j=0}^{r-1} \frac{\lambda^j j w^{j-1} e^{-\lambda w}}{j!} - \sum_{j=0}^{r-1} \frac{\lambda^j w^j (-\lambda) e^{-\lambda w}}{j!} \\ &= -\sum_{j=1}^{r-1} \frac{\lambda^j w^{j-1} e^{-\lambda w}}{(j-1)!} + \sum_{j=0}^{r-1} \frac{\lambda^{j+1} w^j e^{-\lambda w}}{j!} \qquad [\text{set } k = j-1] \\ &= -\sum_{k=0}^{r-2} \frac{\lambda^{k+1} w^k e^{-\lambda w}}{k!} + \sum_{j=0}^{r-1} \frac{\lambda^{j+1} w^j e^{-\lambda w}}{j!} \\ &= \frac{\lambda^r w^{r-1} e^{-\lambda w}}{(r-1)!} = \frac{\lambda^r w^{r-1} e^{-\lambda w}}{\Gamma(r)}, \qquad w \geqslant 0 \end{aligned}$$

i.e. $W_r \sim \text{gamma}(r, \lambda)$.

(ii) (a) $T_n = X_1 + \cdots + X_n \sim \text{gamma}(n, \lambda)$.

(b)



$N(t) \geqslant k \iff k$, or $k+1$, or ... events in $(0, t]$.
$T_k \leqslant t \iff k$, or $k+1$, or ... events in $(0, t]$.
So $N(t) \geqslant k \iff T_k \leqslant t$.

(c) We have

$$\begin{aligned} \mathrm{P}(N(t) \geqslant k) &= \mathrm{P}(T_k \leqslant t) = F_{T_k}(t) \\ &= \mathrm{P}(W \geqslant k) \quad \text{where} \quad W \sim \text{Poisson}(\lambda t) \\ &= \sum_{j=k}^{\infty} \frac{(\lambda t)^j e^{-\lambda t}}{j!}. \end{aligned}$$

/continued overleaf

So

$$\begin{aligned}
\mathrm{P}(N(t)=k) &= \mathrm{P}(N(t)\geqslant k)-\mathrm{P}(N(t)\geqslant k+1)\\
&= \sum_{j=k}^{\infty}\frac{(\lambda t)^j e^{-\lambda t}}{j!}-\sum_{j=k+1}^{\infty}\frac{(\lambda t)^j e^{-\lambda t}}{j!}\\
&= \frac{(\lambda t)^k e^{-\lambda t}}{k!},\quad k=0,1,2,...
\end{aligned}$$

So $N(t)\sim$ Poisson (λt) or $\{N(t),t\geqslant 0\}$ is a Poisson process.

2. (i) The MGF of $X(t)$ (with $X(0)=1$) can be expanded as follows:

$$\begin{aligned}
G(s,t) &= \frac{s-\alpha t(s-1)}{1-\alpha t(s-1)}=\frac{\alpha t+(1-\alpha t)s}{(1+\alpha t)-\alpha ts}\\
&= \frac{\alpha t+(1-\alpha t)s}{(1+\alpha t)\left\{1-\left(\frac{\alpha t}{1+\alpha t}\right)s\right\}}\\
&= \frac{1}{(1+\alpha t)}\{\alpha t+(1-\alpha t)s\}\left\{1+\left(\frac{\alpha t}{1+\alpha t}\right)s+\left(\frac{\alpha t}{1+\alpha t}\right)^2 s^2+\cdots\right\}.
\end{aligned}$$

The constant (s^0) term in this expansion gives

$$p_0(t)=\frac{\alpha t}{1+\alpha t}.$$

For $n\geqslant 1$, the coefficient of s^n gives

$$p_n(t)=\left(\frac{\alpha t}{1+\alpha t}\right)\left(\frac{\alpha t}{1+\alpha t}\right)^n+\left(\frac{1-\alpha t}{1+\alpha t}\right)\left(\frac{\alpha t}{1+\alpha t}\right)^{n-1}=\frac{(\alpha t)^{n-1}}{(1+\alpha t)^{n+1}}.$$

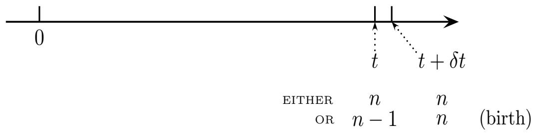
Now

$$\frac{\partial G(s,t)}{\partial s}=\frac{(1-\alpha t)}{(1+\alpha t)-\alpha ts}-\frac{\alpha t+(1-\alpha t)s}{\{(1+\alpha t)-\alpha ts\}^2}(-\alpha t),$$

so

$$\mathrm{E}(X(t))=\left(\frac{\partial G(s,t)}{\partial s}\right)_{s=1}=\frac{1-\alpha t}{1}-\frac{1}{1^2}(-\alpha t)=1.$$

(ii) Pure birth process



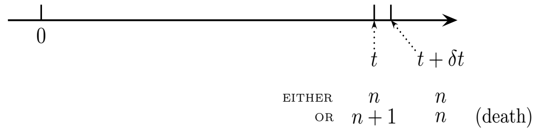
$$\begin{aligned}
p_n(t+\delta t) &= p_n(t)(1-\alpha_n\delta t+o(\delta t))+p_{n-1}(t)(\alpha_{n-1}\delta t+o(\delta t))+o(\delta t),\quad n>a\\
p_a(t+\delta t) &= p_a(t)(1-\alpha_a\delta t+o(\delta t)).
\end{aligned}$$

So

$$\begin{aligned}
\frac{dp_n(t)}{dt} &= -\alpha_n p_n(t)+\alpha_{n-1}p_{n-1}(t),\quad n>a\\
\frac{dp_a(t)}{dt} &= -\alpha_a p_a(t).
\end{aligned}$$

/continued overleaf

(iii) <u>Pure death process</u>

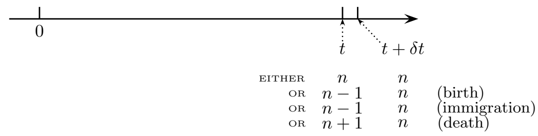


$$
\begin{aligned}
p_n(t+\delta t) &= p_n(t)(1-\beta_n\delta t+o(\delta t)) \\
&\quad +p_{n+1}(t)(\beta_{n+1}\delta t+o(\delta t))+o(\delta t), \quad 0\leqslant n\leqslant a-1 \\
p_a(t+\delta t) &= p_a(t)(1-\beta_a\delta t+o(\delta t)).
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{dp_n(t)}{dt} &= -\beta_n p_n(t)+\beta_{n+1}p_{n+1}(t), \quad 0\leqslant n\leqslant a-1 \\
\frac{dp_a(t)}{dt} &= -\beta_a p_a(t).
\end{aligned}
$$

(iv) <u>Birth and death plus immigration process</u>



$$
\begin{aligned}
p_n(t+\delta t) &= p_n(t)\{1-\alpha_n\delta t-\beta_n\delta t-\theta\delta t+o(\delta t)\} \\
&\quad +p_{n-1}(t)\{\alpha_{n-1}\delta t+o(\delta t)\}+p_{n-1}(t)\{\theta\delta t+o(\delta t)\} \\
&\quad +p_{n+1}(t)\{\beta_{n+1}\delta t+o(\delta t)\}+o(\delta t), \quad n\geqslant 1 \\
p_0(t+\delta t) &= p_0(t)\{1-\alpha_0\delta t-\theta\delta t+o(\delta t)\} \\
&\quad +p_1(t)\{\beta_1\delta t+o(\delta t)\}+o(\delta t).
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{dp_n(t)}{dt} &= -(\alpha_n+\beta_n+\theta)p_n(t)+(\alpha_{n-1}+\theta)p_{n-1}(t)+\beta_{n+1}p_{n+1}(t), \quad n\geqslant 1 \\
\frac{dp_0(t)}{dt} &= -(\alpha_0+\theta)p_0(t)+\beta_1 p_1(t).
\end{aligned}
$$

3. (i) We have

$$
\begin{aligned}
\mathrm{P}(U>u) &= \mathrm{P}(\min(Y_1,Y_2)>u) \\
&= \mathrm{P}(Y_1>u, Y_2>u) \\
&= \mathrm{P}(Y_1>u).\mathrm{P}(Y_2>u) \qquad \text{[since } Y_1, Y_2 \text{ are independent]} \\
&= e^{-\lambda_1 u}.e^{-\lambda_2 u}=e^{-(\lambda_1+\lambda_2)u}, \quad u\geqslant 0.
\end{aligned}
$$

So $\mathrm{P}(U\leqslant u)=1-e^{-(\lambda_1+\lambda_2)u}, \quad u\geqslant 0$

i.e. $U\sim$ negative exponential$(\lambda_1+\lambda_2)$.

<u>Generalization</u>: Consider n types of events, and let

$$Y_i = \text{time to an event of type } i, \qquad i=1,...,n.$$

If the $\{Y_i\}$ are independent and $Y_i\sim$ negative exponential(λ_i), $i=1,...,n$, then

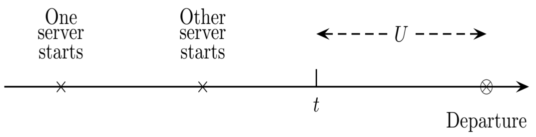
$$U=\min(Y_1,Y_2,...,Y_n)\sim \text{negative exponential}(\sum_{i=1}^{n}\lambda_i)$$

/continued overleaf

Consider a time t at which both servers are serving. Let

Y_i = time from t to end of service by server i

U = time from t to next service completion (customer departure).



Then $U = \min(Y_1, Y_2)$, where, by the no-memory property, Y_1 and Y_2 are negative exponentially distributed (with parameter λ) and independent. Hence $U \sim$ negative exponential(2λ), i.e. the departure rate is 2λ.

If only one server is operating at time t, the departure rate is λ.

(ii) We have

$$(\lambda_n + \mu_n)\pi_n = \lambda_{n-1}\pi_{n-1} + \mu_{n+1}\pi_{n+1}, \quad n \geqslant 0 \tag{1}$$

where $\lambda_{-1} = \mu_0 = 0; \qquad \sum_{n=0}^{\infty} \pi_n = 1.$ (2)

We follow the method given in lectures for the birth and death process.

Summing (1) from $n = 0$ to $n = m$, we get

$$\begin{aligned}
\sum_{n=0}^{m} (\lambda_n + \mu_n)\pi_n &= \sum_{n=0}^{m} \lambda_{n-1}\pi_{n-1} + \sum_{n=0}^{m} \mu_{n+1}\pi_{n+1} \\
&= \sum_{n=0}^{m-1} \lambda_n\pi_n + \sum_{n=0}^{m+1} \mu_n\pi_n \quad [\text{using } (2a)] \\
\text{i.e.} \quad \mu_{m+1}\pi_{m+1} &= \lambda_m\pi_m, \quad m \geqslant 0 \\
\text{i.e.} \quad \pi_{m+1} &= \frac{\lambda_m}{\mu_{m+1}}\pi_m \longrightarrow \frac{\lambda_m \ldots . \lambda_0}{\mu_{m+1} \ldots . \mu_1}\pi_0.
\end{aligned}$$

The normalization requirement $\sum_{m=0}^{\infty} \pi_m = 1$ yields

$$\pi_0 = S^{-1},$$

where

$$S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots$$

provided $\lambda_0, \lambda_1, \ldots$ and $\mu_1, \mu_2, \ldots$ are such that this series converges.

Particular cases:

(a) Queue with discouragement (and constant service rate)

$$\lambda_n = \frac{\lambda}{n+1}, \quad n \geqslant 0 : \qquad \mu_n = \mu, \quad n \geqslant 1.$$

Then

$$\begin{aligned}
S &= 1 + \tfrac{\lambda}{1} \cdot \tfrac{1}{\mu} + \tfrac{\lambda}{1} \cdot \tfrac{\lambda}{2} \cdot \left(\tfrac{1}{\mu}\right)^2 + \tfrac{\lambda}{1} \cdot \tfrac{\lambda}{2} \cdot \tfrac{\lambda}{3} \cdot \left(\tfrac{1}{\mu}\right)^3 + \cdots \\
&= 1 + \left(\tfrac{\lambda}{\mu}\right) + \tfrac{1}{2!}\left(\tfrac{\lambda}{\mu}\right)^2 + \tfrac{1}{3!}\left(\tfrac{\lambda}{\mu}\right)^3 + \cdots \\
&= e^{(\lambda/\mu)}.
\end{aligned}$$

So

$$\begin{aligned}
\pi_0 &= e^{-\lambda/\mu} \\
\pi_n &= \tfrac{1}{n!}(\lambda/\mu)^n e^{-\lambda/\mu}, \qquad n \geqslant 1.
\end{aligned}$$

/continued overleaf

Note:
(i) There is no restriction on the values of λ and μ, other than $\lambda, \mu \geqslant 0$.
(ii) We observe that $\{\pi_n\}$ is a Poisson distribution with parameter λ/μ.

(b) M/M/s queue

$$\lambda_n = \lambda, \quad n \geqslant 0: \qquad \mu_n = \begin{cases} n\mu, & n \leqslant s \\ s\mu, & n > s. \end{cases}$$

Then

$$\begin{aligned} S &= 1 + \tfrac{\lambda}{\mu} + \tfrac{\lambda^2}{\mu.2\mu} + \cdots + \tfrac{\lambda^s}{\mu.2\mu....s\mu} + \tfrac{\lambda^s}{s!\mu^s} \sum_{j=1}^{\infty} \tfrac{\lambda^j}{(s\mu)^j} \\ &= 1 + \left(\tfrac{\lambda}{\mu}\right) + \tfrac{1}{2!}\left(\tfrac{\lambda}{\mu}\right)^2 + \cdots + \tfrac{1}{s!}\left(\tfrac{\lambda}{\mu}\right)^s + \tfrac{\lambda^s}{s!\mu^s} \cdot \left(\tfrac{\lambda}{s\mu}\right) \cdot \tfrac{1}{1-\left(\frac{\lambda}{s\mu}\right)} \quad (\lambda < s\mu) \\ &= 1 + \left(\tfrac{\lambda}{\mu}\right) + \tfrac{1}{2!}\left(\tfrac{\lambda}{\mu}\right)^3 + \cdots + \tfrac{1}{(s-1)!}\left(\tfrac{\lambda}{\mu}\right)^{s-1} + \tfrac{\lambda^s}{s!\mu^s}\left\{1 + \left(\tfrac{\lambda}{s\mu}\right) \cdot \frac{1}{1-\left(\frac{\lambda}{s\mu}\right)}\right\} \\ &= 1 + \left(\tfrac{\lambda}{\mu}\right) + \cdots + \tfrac{1}{(s-1)!}\left(\tfrac{\lambda}{\mu}\right)^{s-1} + \tfrac{\lambda^s}{s!\mu^s} \cdot \frac{1}{1-\left(\frac{\lambda}{s\mu}\right)}. \end{aligned}$$

Provided $\lambda < s\mu$, the steady-state distribution is

$$\pi_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n S^{-1}, & n \leqslant s \\ \frac{s^s}{s!}\left(\frac{\lambda}{s\mu}\right)^n S^{-1}, & n > s \end{cases}$$

$\nearrow$

$$\text{or } \left(\frac{\lambda}{s\mu}\right)^j \pi_s, \qquad n = s + j.$$

(c) Queue with constant arrival rate and ‘ample’ servers

$$\lambda_n = \lambda, \quad n \geqslant 0; \qquad \mu_n = n\mu, \quad n \geqslant 1.$$

Then

$$\begin{aligned} S &= 1 + \tfrac{\lambda}{\mu} + \tfrac{\lambda^2}{\mu.2\mu} + \tfrac{\lambda^3}{\mu.2\mu.3\mu} + \cdots \\ &= 1 + \left(\tfrac{\lambda}{\mu}\right) + \tfrac{1}{2!}\left(\tfrac{\lambda}{\mu}\right)^2 + \tfrac{1}{3!}\left(\tfrac{\lambda}{\mu}\right)^3 + \cdots = e^{\lambda/\mu} \end{aligned}$$

So $\quad S^{-1} = e^{\lambda/\mu} \quad$ and

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu}, \qquad n \geqslant 0.$$

(d) Queue with constant service rate and limited capacity

$$\lambda_n = \begin{cases} \lambda, & n < N \\ 0, & n \geqslant N \end{cases} : \qquad \mu_n = \mu, \quad n \geqslant 1.$$

Then

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} + \cdots + \frac{\lambda^N}{\mu^N} = \frac{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}{1 - \frac{\lambda}{\mu}}$$

and

$$\pi_n = \frac{\left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}, \qquad n = 0, 1, ..., N.$$

— — — — —