Chapter 8

Interest rate derivatives

A bond is a long-term contract under which the issuer (or borrower) promises to pay the bondholder *coupon interest payments* (usually periodic) and *principal* on specific dates as stated in the bond indenture. If there is no coupon payment during the life of the bond, the bond is said to be a *zero-coupon bond*. A bond is a common financial instrument used by firms or governments to raise capital, whereby the up front premium paid by the bond holders can be considered as a loan to the issuer. The face value of the bond is called the *par value* and the *maturity date* of the bond is the specified date on which the par value of a bond must be repaid.

The risk-free interest rate is assumed constant in most of the pricing models, for example, the Black-Schole model. Such an assumption is acceptable when the life of a financial derivative is short (upto six to nine months). Because the life span of a bond is usually 10 years or longer, it is unrealistic to assume the interest rate to remain constant throughout the whole life of the bond. After the bond is being launched, the value of the bond changes over time until maturity due to the change in its life span. Recently, we have witnessed a proliferation of financial derivatives based on interest rates, for example, bond futures, options on bonds, swaps and bonds with option features. The value of these interest-rate derivate products depend sensibly on the level of interest rates.

Table 8.1: Spot and Forward Interest Rates

Year (n)	Spot rate for an n-year investment	Forward rate for nth year
1	10.0%	
2	10.5%	11.0%
3	10.8%	11.4%
4	11.0%	11.6%
5	11.1%	11.5%

8.1 Calculation of forward rates

8.1.1 Spot and forward interest rates

The spot interest rate is the interest rate appropriate for a one-period loan. This loan is fixed today.

Let us consider a simple example. Somebody wants to invest £1,000 for 2 years. Two strategies open to him. Strategy 1 is to put the money in a 1-year zero-coupon bond at an interest rate r_1. At the end of the year, he takes his money and finds another 1-year bond. Let us call the rate of interest on this second bond ${}_1r_2$-that is, the spot rate of interest at time 1 on a loan maturing at time 2. The final payoff to this strategy is

$$£1,000\mathrm{e}^{r_1 + {}_1r_2} \tag{8.1}$$

Instead of making two separate investments of 1 year each, he could invest his money today in a zero-coupon bond that pays off in year 2. If the 2-year spot interest rate is r_2, the final payoff is

$$£1,000\mathrm{e}^{2r_2} \tag{8.2}$$

Of course, he cannot know for sure what the one-period spot rate of interest ${}_1r_2$ will be next year. The first strategy can be interpreted as investing for 1 year at the spot rate r_1 and for the second year at a *forward rate* f_2. This forward rate is implicit in the 2-year spot rate r_2 and it is also guaranteed. By buying the 2-year bond, he can "lock in" an interest rate of f_2 for the second year. Using Eqs.(8.1) and (8.2) for ${}_1r_2 = f_2$, we find that

$$f_2 = 2r_2 - r_1. \quad (8.3)$$

Spot rates and forward rates are compared in Table 8.1. In general, if r is the spot rate of interest applying for T years and r^* is the spot rate of interest applying for T^* years where $T^* > T$, the forward interest rate for the period of time between T and T^*, f, is given by

$$f = \frac{r^*T^* - rT}{T^* - T} = r^* + T\frac{r^* - r}{T^* - T}. \quad (8.4)$$

If $r^* > r$ (upward sloping), then $f > r^* > r$ so that forward rates are higher than zero-coupon yields. On the other hand, if $r^* < r$ (downward sloping), then $f < r^*$ so that zero-coupon yields are higher than forward rates. This is shown in Fig. 8.1.1. In the limit $T^* \to T$, we obtain the instantaneous forward rate for a maturity T:

$$f_{inst} = \lim_{T^* \to T, r^* \to r} f = r + T\frac{\partial r}{\partial T}. \quad (8.5)$$

8.1.2 Bootstrap method

In the market, we have the prices of coupon-bearing bonds from which we have to extract the zero-coupon yields. We use the *bootstrap method* for this purpose.

Let us assume that we observe the data shown in Table 8.2. The calculation of continuous compounding rates for three months, six months and one year is straightforward using Eqs. (2.3). Consider the fist bond type shown with maturity in 3 months. Here $A_n/A_o = 100/97.5$. Thus the three-month rate with continuous compounding, $r_{0.25}$, is

$$r_{0.25} = 4\ln\left(\frac{100}{97.5}\right) = 0.1013 \quad (8.6)$$

The yield is = 10.13% per annum. Similarly, the six-month rate is 10.47% per annum and the one-year rate is 10.54% per annum.

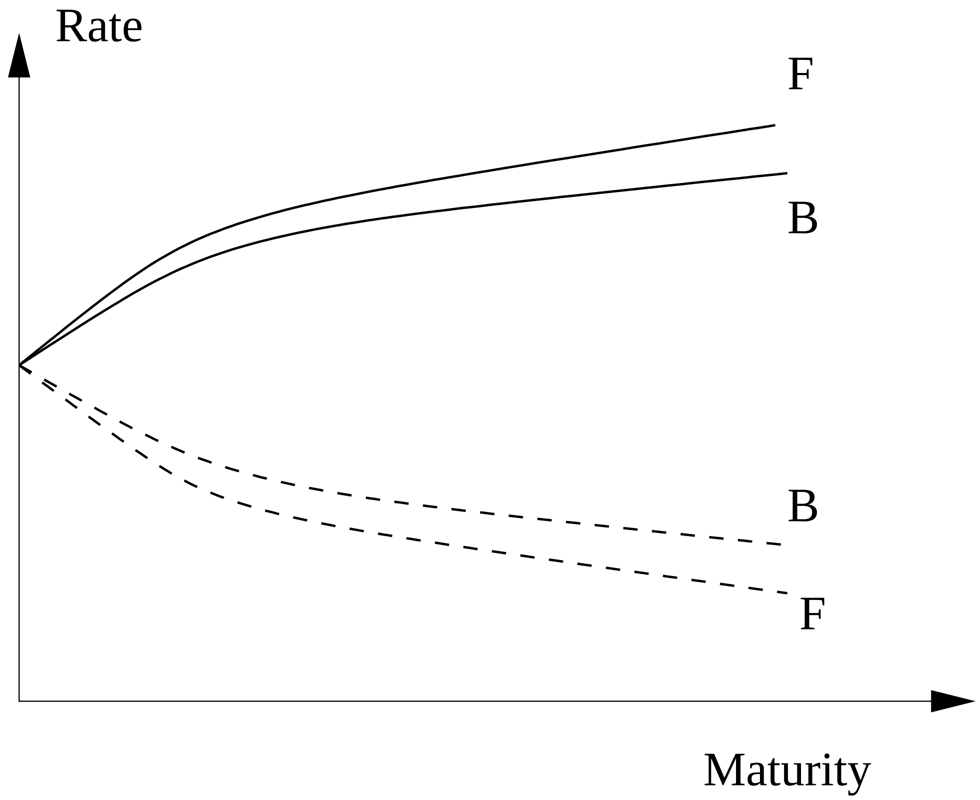

Figure 8.1: Forward rate (F) and zero-coupon yield (B) for upward sloping (solid-line) and downward sloping

Table 8.2: Bond prices (Half of the coupon is assumed to be paid every six months)

Bond principal(£)	Time to maturity (years)	Annual coupon(£)	Bond price(£)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6
100	2.75	10	99.8

The payments for the fourth bond is

$$\begin{aligned} 6 \text{ months} &: \text{£}4 \\ 1 \text{ year} &: \text{£}4 \\ 1.5 \text{ years} &: \text{£}104 \end{aligned}$$

We know that the discount rate for the payment at the end of six months (one year) is 10.47% (10.54%). We denote the 1.5-year spot rate by R then from

$$4\mathrm{e}^{-0.1047\times 0.5} + 4\mathrm{e}^{-0.1054\times 1.0} + 104\mathrm{e}^{-R\times 1.5} = 96 \tag{8.7}$$

we find the 1.5-year spot rate 10.68%. Similarly two-year spot rate is 10.81%.

The payments for the sixth bond is

$$\begin{aligned} 3 \text{ months} &: \text{£}5 \\ 9 \text{ months} &: \text{£}5 \\ 1.25 \text{ years} &: \text{£}5 \\ 1.75 \text{ years} &: \text{£}5 \\ 2.25 \text{ years} &: \text{£}5 \\ 2.75 \text{ years} &: \text{£}105 \end{aligned}$$

Using linear interpolation, we find the 9-month, 1.25-year and 1.75-year spot rates 10.505, 10.61 and 10.745%, respectively. If the 2.75-year spot rate is R', then using linear interpolation, the 2.25-year spot rate is $0.1081 \times (2/3) + R'/3$. Therefore, we have

$$\begin{aligned} 5\mathrm{e}^{-0.1013\times0.25} & +5\mathrm{e}^{-0.10505\times0.75} + 5\mathrm{e}^{-0.1061\times1.25} + 5\mathrm{e}^{-0.10745\times1.75} \\ & +5\mathrm{e}^{-(0.0721+R'/3)\times2.25} + 105\mathrm{e}^{-R'\times2.75} = 99.8 \end{aligned}$$

Using a numerical procedure, we find $R = 0.1087$ (10.87% per annum).

8.1.3 Day count conventions

There are three day count conventions used in practice:

1. actual/actual (in period)
2. 30/360
3. actual/360.

The first is straightforward. The second considers 360 days in 1 year and 30 days in 1 month. The third considers 360 days a year.

8.2 Forward rate agreement

Since 1983, banks have been prepared to enter into forward contracts to borrow or lend money. If a company buys one of these *forward rate agreements* (FRAs), it agrees to borrow in the future at a rate that is fixed today; if the company sells an FRA, it agrees to lend in the future at a preset rate.

Consider an FRA where it is agreed at time $t = 0$ that an interest rate of R_K will be earned for the period of time between T and T^* on a principal of £100. The FRA incurs the following cash flows:

$$\begin{aligned} \text{Time } T &: -\pounds 100 \\ \text{Time } T^* &: +\pounds 100 \exp[R_K(T^* - T)] \end{aligned}$$

Suppose

$$\begin{aligned} r &: \text{ spot interest for maturity } T \\ r^* &: \text{ spot interest rate for maturity } T^* \end{aligned} \tag{8.8}$$

The value, $V(0)$, of the agreement at time $t = 0$ is

$$V(0) = £100\mathrm{e}^{R_K(T^*-T)}\mathrm{e}^{-r^*T^*} - £100\mathrm{e}^{-rT} \tag{8.9}$$

which is zero when

$$R_K = \frac{r^*T^* - rT}{T^* - T} \tag{8.10}$$

Comparing this with Eq.(8.4) we find that the agreed rate in an FRA should be the same as the forward rate f_o at the time the contract is initiated.

If R is the interest rate at time T for the period between T and T^*, the present value of the FRA at time $t = T$ is

$$£100\mathrm{e}^{R_K(T^*-T)}\mathrm{e}^{-R(T^*-T)} - £100.$$

If the spot interest rate is r_t (r_t^*) at time t ($0 \le t \le T$) for the maturity T (T^*), the value of the deal at t is

$$\begin{aligned} V(t) &= £100\mathrm{e}^{R_K(T^*-T)}\mathrm{e}^{-r_t^*(T^*-t)} - £100\mathrm{e}^{-r_t(T-t)} \\ &= [£100\mathrm{e}^{(R_K-f_t)(T^*-T)} - £100]\mathrm{e}^{-r_t(T-t)} \end{aligned} \tag{8.11}$$

where f_t is the forward rate at time t

$$f_t = \frac{r_t^*(T^*-t) - r_t(T-t)}{T^* - T}.$$

When $t = 0$, $f_o = R_K$ and the value of the FRA becomes zero. Eq.(8.11) shows that we can value FRAs by calculating the present value of cash flows on the assumption that the current forward rates are realised.

8.3 Treasury bill futures

In the Treasury bill futures contract, the underlying asset is a Treasury bill[1]. For example, consider a 90-day Treasury bill futures. The party with the short position must deliver Treasury bills which may have 91 days to expiration. The party can choose to deliver it on the next day or two days later in which case the Treasury bills have 90 or 89 days to mature, respectively. Prior to maturity of the futures contract, the underlying asset can be considered as a Treasury bill with a maturity longer than 90 days. For example, if the futures contract matures in 80 days, the underlying asset is a 170-day Treasury bill.

We want to calculate the value of a Treasury bill futures contract which matures at T and the Treasury bill underlying the futures contract matures in T^*. If the Treasury bill underlying the futures contract has a face value of \$100, its current value, V^*, is given by

$$V^* = 100\mathrm{e}^{-r^* T^*} \tag{8.12}$$

where r and r^* are defined in (8.8).

The futures price, F, is

$$F = 100\mathrm{e}^{rT - r^* T^*} = 100\mathrm{e}^{-f(T^* - T)}. \tag{8.13}$$

8.4 Eurodollar futures

A Eurodollar is a dollar deposited in a bank outside the United States. The Eurodollar interest rate is the rate of interest earned on Eurodollars deposited by a bank with another bank. This is also known as the London Interbank Offer Rate (LIBOR). Eurodollar interest rates are generally higher than the corresponding Treasury bill interest rates.

The Eurodollar futures contract is a popular contract traded, for example, on the Chicago Mercantile Exchange (CME).

[1] A Treasury bill pays no coupons and the investor receives the face value at maturity

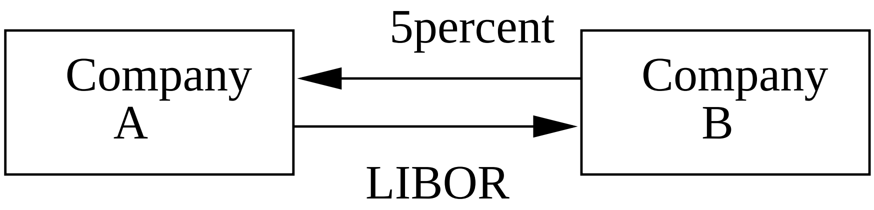

Figure 8.2: Interest rate swap

8.5 Swap

An interest rate *swap* is an agreement between two parties to exchange the interest rate payments on a certain amount (*principal*), for certain length of time. For example, one party pays the other a fixed rate of interest in return for a floating interest rate payment.

LIBOR LIBOR is the London Interbank Offer Rate which is the interest rate at which major international banks in London lend. LIBOR changes continuously as economic conditions change. LIBOR is frequently a reference rate of interest for loans in international financial markets.

Let us consider a three-year swap initiated on 1 March 1998. Company B pays to company A a rate of 5% per annum on a notional principal of £100 million and company A pays company B the six-month LIBOR on the same notional principal. They agreed that payments are to be exchanged every six months and the 5% interest rate is quoted with *semiannual compounding.* The agreement is schematically shown in Fig. 8.2 and the payment schedule and net cash flow are shown in Table 8.2

Normally the £100 million principal is used only for the calculation of interest payments. The principal itself is not exchanged so it is called the *notional principal.* A swap can be characterized as the difference between two bonds. Although the principal is not exchanged, we can assume without changing the value of the swap that at the end of its life the principals are exchanged. Under the assumption, company B's position is described as long a floating-rate bond and short a

Table 8.3: Cash flows to company B (£ millions)

Date	LIBOR(%)	Floating cash flow	Fixed cash flow
1 Mar. 1998	4.20		
1 Sep. 1998	4.80	+2.10	-2.50
1 Mar. 1999	5.30	+2.40	-2.50
1 Sep. 1999	5.50	+2.65	-2.50
1 Mar. 2000	5.60	+2.75	-2.50
1 Sep. 2000	5.90	+2.80	-2.50
1 Mar. 2001	6.40	+2.95	-2.50

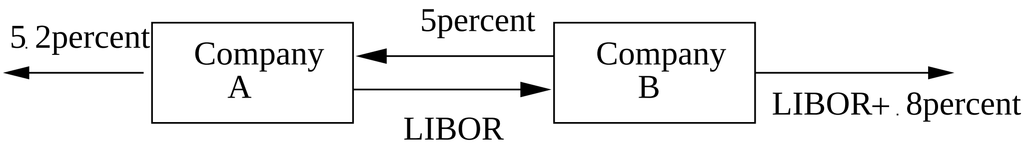

Figure 8.3: Swap agreement to transform a liability

fixed-rate bond and company A's position is described as long a fixed-rate bond and short a floating-rate bond.

The swap can be used

· to transform a liability

When the swap is agreed as shown in Fig. 8.3, for company A the swap has the effect of transforming borrowings at a fixed rate of 5.2% into borrowings at a floating rate of LIBOR plus 20 basis points (0.2%).

· to transform an asset

When the swap is agreed as shown in Fig. 8.4, for company A the swap is to transform an asset earning LIBOR minus 25 basis points into an asset earning 4.75%.

To avoid the risk of default for either party, it is common to have an interme-

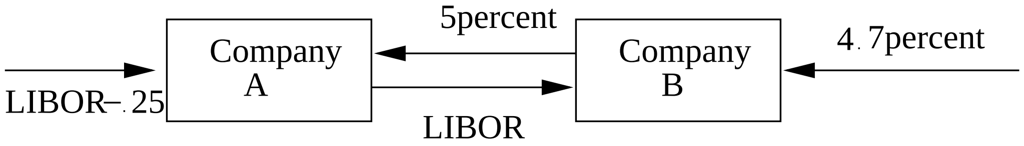

Figure 8.4: Swap agreement to transform an asset

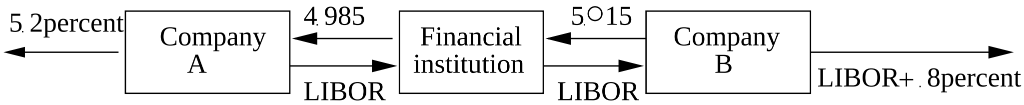

Figure 8.5: Role of a financial institution for the swap agreement to transform a liability shown in Fig. 8.3.

diary in the swap transaction. The companies each deal with a financial intermediary such as a bank or other financial institution. As shown in Fig.8.5, the financial institution earns about 3 basis points (= 0.03%) on a pair of offsetting transactions.

Valuation of interest rate swaps

Assuming that the principal is £100 million the swap between company B and the financial institution shown in Fig.8.5 is

1. Company lends the financial institution £100 million at the six-month LIBOR
2. The financial institution lends company B £100 million at a fixed rate of 5.015% per annum.

The agreement is interpreted in this way, the financial institution has sold a £100 million floating rate (LIBOR) bond to company in return to the purchase of a £100 million fixed-rate (5.015% per annum) bond from company B. The value, V, of the swap is thus

$$V = B_{fix} - B_{fl} \tag{8.14}$$

where B_{fix} is the value of fixed-rate bond that pays k at time t_i and the principal amount of Q at time t_n:

$$B_{fix} = \sum_{i=1}^{n} k\mathrm{e}^{-r_i t_i} + Q\mathrm{e}^{-r_n t_n} \tag{8.15}$$

and B_{fl} is the value of floating-rate bond:

$$B_{fl} = k^* \mathrm{e}^{-r_1 t_1} + Q\mathrm{e}^{-r_1 t_1} \tag{8.16}$$

with k^* the floating-rate payment which will be payed at t_1. The floating-rate payment at t_1 is already known because it is the LIBOR at the last payment date (not the LIBOR at the actual payment date).

Example: Suppose that under the terms of a swap, Company A has agreed to receive six-month LIBOR and pay a fixed interest of 10% per annum with semiannual payment. The swap has a remaining life of fifteen months. The six-month LIBOR at the last payment date was 12%. The continuous-compounding interest rates are 9%, 10% and 11% for the maturities of three months, nine months and fifteen months, respectively. What is the value of the swap?

8.6 Interest rate models

The bond price is normally a function of interest rate and time. We will find the govorning equation for the price of a bond for the two cases of time dependence for the interest rate: 1) The interest rate is *a known function of time* 2) The interest rate is a stochastic variable following Brownian motion.

8.6.1 Bond pricing with determinsistic interest rates

Consider that the interest rate $r(t)$ is not an independent state variable but itself is a known function of time. The bond price is, thus, simply a function of time. Let

$$\begin{aligned} B(t) &: \text{ Bond price} \\ k(t) &: \text{ continuos coupon.} \end{aligned}$$

The final condition of $B(t)$ is $B(T) = F$ where F is the principal of the bond and T is the maturity time of the bond.

Suppose a portfolio composed of a long position in one bond. The change in the value of the portfolio is

$$dB + k(t)dt \tag{8.17}$$

Under the arbitrage-free assumption, this portfolio should earn risk-free interest because the portfolio is a known function of time and risk-free. Thus

$$dB + k(t)dt = r(t)Bdt \tag{8.18}$$

Solving this equation under the final condition $B(T) = F$, we find

$$B(t) = \mathrm{e}^{-\int_t^T r(s)ds} \left[F + \int_t^T k(\tau)\mathrm{e}^{\int_\tau^T r(s)ds} d\tau \right]. \tag{8.19}$$

The first term in the right hand side of the equation corresponds to the present value of the principal and the second term to the present value of the coupon stream.

8.6.2 One-factor bond pricing model

We derive a governing equation for a zero-coupon bond when the interest rate is considered as a stochastic variable following the Brownian motion.

Suppose the spot rate $r(t)$ follows the Brownian motion

$$dr = u(r,t)dt + w(r,t)dZ \tag{8.20}$$

where dZ is the standard Wiener process, $u(r,t)$ and $w(r,t)^2$ are the instantaneous drift and variance of the process for $r(t)$. We assume that the bond price depends only on the spot interest rate r, current time t and maturity time T. Using Ito's lemma the dynamics of the bond price is written as

$$\frac{dB}{B} = \mu_B(r,t)dt + \sigma_B(r,t)dZ \tag{8.21}$$

where the drift and the volatility are, respectively,

$$\begin{aligned} \mu_B(r,t) &= \frac{1}{B}\left(\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2}\right) \\ \sigma_B(r,t) &= \frac{w}{B}\frac{\partial B}{\partial r}. \end{aligned} \quad (8.22)$$

Because the interest rate is not a traded security, it cannot be used to hedge the bond so that we try to hedge bonds of different maturities. Consider the portfolio composed of a long position in a bond of dollar value V_1 with maturity T_1 and a short position in another bond of dollar value V_2 with maturity T_2. The value of the portfolio is

$$\Pi = V_1 - V_2. \quad (8.23)$$

With use of Eq.(8.21), the change in portfolio value in time dt is

$$d\Pi = [V_1\mu_B(r,t,T_1) - V_2\mu_B(r,t,T_2)]dt + [V_1\sigma_B(r,t,T_1) - V_2\sigma_B(r,t,T_2)]dZ. \quad (8.24)$$

If we choose

$$V_1 = \frac{\sigma_B(r,t,T_2)}{\sigma_B(r,t,T_2) - \sigma_B(r,t,T_1)}\Pi \text{ and } V_2 = \frac{\sigma_B(r,t,T_1)}{\sigma_B(r,t,T_2) - \sigma_B(r,t,T_1)}\Pi \quad (8.25)$$

then the stochastic term in Eq. (8.24) vanishes and the equation becomes

$$d\Pi = \frac{\mu_B(r,t,T_1)\sigma_B(r,t,T_2) - \mu_B(r,t,T_2)\sigma_B(r,t,T_1)}{\sigma_B(r,t,T_2) - \sigma_B(r,t,T_1)}\Pi dt. \quad (8.26)$$

This risk-free portfolio should earn the risk-free interest, *i.e.*, $d\Pi = r(t)\Pi dt$, thus using Eqs.(8.23) and (8.26),

$$\begin{aligned} & \frac{\mu_B(r,t,T_1)\sigma_B(r,t,T_2) - \mu_B(r,t,T_2)\sigma_B(r,t,T_1)}{\sigma_B(r,t,T_2) - \sigma_B(r,t,T_1)}\Pi dt = r(t)\Pi dt \\ \Rightarrow \quad & \frac{\mu_B(r,t,T_1) - r(t)}{\sigma_B(r,t,T_1)} = \frac{\mu_B(r,t,T_2) - r(t)}{\sigma_B(r,t,T_2)} \end{aligned} \quad (8.27)$$

This relation is valid for arbitrary maturity dates T_1 and T_2 so that the ratio $\frac{\mu_B(r,t,T) - r(t)}{\sigma_B(r,t,T)}$ should be independent of maturity T. We denote the ratio by $\lambda(r,t)$ and call it the *market price of risk*:

$$\lambda(r,t) = \frac{\mu_B(r,t,T) - r(t)}{\sigma_B(r,t,T)} \quad (8.28)$$

Substituting μ_B and σ_B in Eqs.(8.22) into Eq.(8.28) we find the following governing equation for the price of a zero-coupon bond

$$\frac{\partial B}{\partial t} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} + (u - \lambda w)\frac{\partial B}{\partial r} - rB = 0. \tag{8.29}$$

8.7 European bond options

European bond options are European options on bonds. We extend the Black-Scholes pricing framework to price these bond options.

Suppose we assume the volatility, σ_B, of a pure discount bond to be constant, the Black-Scholes framework can be applied directly for pricing bond options. This is done simply by replacing the current stock price by the current bond price in the Black-Scholes formula for equity options. The value of a European call option on a default-free bond is

$$c(B, \tau) = BN(d_1) - \mathrm{e}^{-R\tau} XN(d_2), \tag{8.30}$$

where $\tau = T - t$ is the remaining time to mature and

$$d_1 = \frac{\ln \frac{B}{X} + \left(R + \frac{\sigma_B^2}{2}\right)\tau}{\sigma_B\sqrt{\tau}} \ , \quad d_2 = d_1 - \sigma_B\sqrt{\tau} \tag{8.31}$$

where B is the current bond price, X is the strike price, R is the current interest rate applicable to a risk-free investment maturing on the expiration date of the option.

8.8 Swaptions and interest rate caps

Swaptions

A swaption provides the holder the right but not the obligation to enter a swap agreement at some time in the future. Let T and T_S denote the expiration dates of the swaption and swap, respectively, $T < T_S$. Let X be the strike price of the swaption. The final condition is given by

$$V(r,T) = \max(\eta(W(r,T) - X), 0) \tag{8.32}$$

where η *is a binary variable which takes the value 1 for a call and -1 for a put.* One has to solve for the swap value $W(r,T)$ first and use it to get the value of a swaption.

Assume the following swap agreement: Company A agrees to pay to company B cash flows equal to interest at a fixed rate r^* on a notional principal for a predetermined period of time. In return, company A receives from company B cash flows equal to interest at a floating rate r on the same notional principal. Although the companies exchange the net difference in the promised interest payments at regular intervals, for convenience, assume continuous cash flows between the two parties. Party A can be considered to receive coupon payment at the rate $r - r^*$ on a simple bond with zero par value at the swap expiration date T_S. If the rate follows the diffusion process discussed earlier, the governing equation for $W(r,t)$ is similar to that of the zero coupon bond except with the addition of the coupon payment term $r - r^*$. Thus

$$\frac{\partial W}{\partial t} + \frac{w^2}{2}\frac{\partial^2 W}{\partial r^2} + (u - \lambda w)\frac{\partial W}{\partial r} - rW + (r - r^*) = 0 \tag{8.33}$$

where $\lambda(r,t)$ is the market price of the risk, and $u(r,t)$ and $w((r,t)$ are the instantatneous drift and variance of the process r. The final condition is

$$W(r, T_s) = 0.$$

Interest rate caps

Interest rate caps are designed to provide insurance against the rate of interest on a floating-rate loan rising above a certain level. This level is known as the *cap rate*. There are two most basic types of interest rate caps:*ceiling* and *floor*.

A ceiling guarantees that the interest rate charged on a floating rate loan at any given time will be the minimum of the prevailing and ceiling rates.

A floor is just the opposite to a ceiling. It guarantees the holder to receive the maximum of the prevailing and floor rates on a floating rate deposit.

A collar agreement specifies both the upper and lower limits for the interest rate that will be charged on a floating rate loan. When the interest rate lies between the limits, the prevailing rate is charged. Otherwise the ceiling (floor) rate is charged when the rate is above (below) the upper (lower) limit.