Low discrepancy sequences generated from permutation polynomials

supervised by

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The main question of uniform distribution theory is: How to distribute an initially unknown number of points as regularly as possible in a given space? The space of interest can be the unit interval $[0, 1]$, the unit square, a sphere or a more abstract space such as a group of rotations etc; see [2]. This question has important applications in an area of numerical integration known as Quasi-Monte Carlo integration.

The aim of this project is to study explicit constructions of finite point sets and infinite sequences. There are two classical constructions of so called low discrepancy sequences, i.e. sequences whose first N points are very regularly distributed. These are permuted van der Corput sequences as well as $(n\alpha)$-sequences. It was recently shown, how to relate the two constructions to each other in a certain way [3, 4] using continued fractions. As a byproduct of this analysis it turned out that permuted van der Corput sequences can significantly improve the best known $(n\alpha)$-sequences. However, these improvements are all based on extensive computer searches for good generating permutations. At the same time the structure of such permutations is not yet understood. In other words, we have large sets of good generating permutations, but we do not understand what makes them good.

Starting from the results in [4] the main goal of this research project is to take a systematic look at the set of all permutations of $\{0, 1, 2, \ldots, p-1\}$ for a prime p using the notion of Carlitz rank; see [5]. This concept is based on the fact that every permutation of $\{0, 1, 2, \ldots, p-1\}$ can be characterised by a specific (permutation) polynomial; see [1]. So far only the distribution properties of sequences generated from permutations of small Carlitz rank are well understood; see [4]. However, preliminary computer experiments suggest that permutations of larger Carlitz rank are the really interesting objects in this context and appear to be the right lens for the study of the structure of good generating permutations.

References

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